

Graph C^* -algebras and the Abelian Core

by

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AUTHOR'S DECLARAION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

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1. INTRODUCTION

Given a directed graph E , we may associate to E a C^* -algebra, $C^*(E)$, by associating edges in E to partial isometries and vertices in E to pairwise orthogonal Hilbert spaces which satisfy some additional conditions. Such graph algebras were first studied by Cuntz and Krieger in 1980 [4]. Because the structure theory of the C^* -algebras is related to the combinatorial and geometrical properties of the underlying graph E , graph algebras have gained a lot of attention. Examples of such graph algebras include some AF-algebras, Cuntz-Krieger algebras, and C^* -algebras built up from matrices over $\mathcal{C}(\mathbb{T})$.

Our report is organized as follows. Section 2 will cover the basic terminology and properties one may associate to the C^* -algebra of the graph E which give us interesting relations between the projections and partial isometries. In order to gain a good grasp on the concepts, we will look at some particular graphs which generate familiar C^* -algebras, including all finite C^* -algebras, the Toeplitz algebra, and $\mathcal{C}(\mathbb{T})$. Section 3 will investigate the conditions which allow us to concretely find $C^*(E)$. We begin with the Gauge-Invariant Uniqueness Theorem and proceed on to an application of this theorem regarding the equality of the C^* -algebras generated by a graph and its so-called dual graph. We conclude this section with the CK-Uniqueness Theorem, which tells us that if every cycle has an entry, every non-degenerate CK E -family generates isomorphic C^* -algebras. These uniqueness theorems also allow us to present a number of graphs whose corresponding C^* -algebras will be familiar to the reader. Section 4 examines the ideals of $C^*(E)$ and completely classifies when the C^* -algebra is simple. Finally, Section 5 is a study on the paper by Nagy and Reznikoff [9]. We begin by introducing the abelian core of a graph C^* -algebra and work towards proving an additional uniqueness theorem says that a $*$ -homomorphism on the C^* -algebra being injective is equivalent to it being injective on the abelian core. As well, a brief investigation into the spectrum of elements in the C^* -algebra give an additional equivalent condition regarding the spectrum of the image of some particular elements.

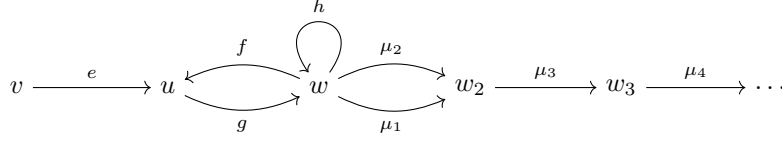
2. BACKGROUND ON GRAPH C^* -ALGEBRAS

2.1. Cuntz-Krieger families. We begin by defining a Cuntz-Krieger family and then examining some examples of Cuntz-Krieger families for given graphs.

Definition 2.1. A **directed graph** E is a collection $E = (E^0, E^1, r, s)$ where E^0 is a set of points, called **vertices**, E^1 can be viewed as a collection of ordered pairs $e = (v, w) \in E^0 \times E^0$ called **edges**, and $r, s : E^1 \rightarrow E^0$ are maps denoting the range and source of an edge, respectively. If a vertex v does not receive edges or equivalently, if $r^{-1}(v) = \emptyset$, then we call the vertex a **source**. We call a vertex v which does not emit edges a **sink**.

The existence of sources will prove to be particularly interesting in many of the theorems presented in this report. We restrict our attention to graphs where each vertex receives finitely many edges and we call such graphs **row-finite**. For the purposes of this report, all graphs will be assumed to be row-finite. To reinforce these new terms, we look at an example.

Example 2.2. Suppose $E^0 = \{u, v, w, w_2, w_3, \dots\}$ and $E^1 = \{e, f, g, h, \mu_1, \mu_2, \mu_3, \dots\}$, where we draw E as in Graph 1.



GRAPH 1. an example row-finite graph

Since v receives no edges, it is a source. If we consider the edge e , then we have $s(e) = v$ and $r(e) = u$. Notice that we allow multiple edges between the same pair of vertices, as demonstrated by the existence of both μ_1 and μ_2 .

Definition 2.3. We let \mathcal{H} be a Hilbert space, E a directed graph, and define a **Cuntz-Krieger E -family** $\{S, P\}$ on \mathcal{H} to consist of a set $P = \{P_v \mid v \in E^0\}$ of mutually orthogonal projections on \mathcal{H} and a set $S = \{S_e \mid e \in E^1\}$ of partial isometries on \mathcal{H} which satisfy

- (CK 1) $S_e^* S_e = P_{s(e)}$ for all $e \in E^1$
- (CK 2) $P_v = \sum_{\{e \in E^1 \mid r(e)=v\}} S_e S_e^*$ whenever v is not a source

We will shorten the terminology by calling such a $\{S, P\}$ a **CK E -family** and throughout this report, we will often consider the C^* -algebra generated by a CK E -family, $C^*(S, P)$.

Note that because S_e is a partial isometry, we can write $S_e = S_e S_e^* S_e = S_e (S_e^* S_e) = S_e P_{s(e)}$. Furthermore, since $S_e \mathcal{H} = S_e S_e^* S_e \mathcal{H} \subseteq S_e S_e^* \mathcal{H}$ (in fact, we have equality here) and $P_{r(e)} = S_e S_e^* + \sum_{\{f \neq e \mid r(f)=r(e)\}} S_f S_f^*$ where $S_f S_f^* \geq 0$, then we have $P_{r(e)} \geq S_e S_e^*$. It follows that for $h \in S_e S_e^* \mathcal{H}$,

$$\|P_{r(e)} h\|^2 = \langle P_{r(e)} h, P_{r(e)} h \rangle = \langle P_{r(e)} h, h \rangle \geq \langle S_e S_e^* h, h \rangle = \langle h, h \rangle = \|h\|^2.$$

Since we can write $\|h\|^2 = \|P_{r(e)} h\|^2 + \|(I - P_{r(e)}) h\|^2$, we get that $\|(I - P_{r(e)}) h\|^2 = 0$ and so $h = P_{r(e)} h \in P_{r(e)} \mathcal{H}$. Thus, $S_e S_e^* \mathcal{H} \subseteq P_{r(e)} \mathcal{H}$, hence for any $h \in \mathcal{H}$, we have $P_{r(e)} S_e h = S_e h$. This gives us the equality

$$S_e = P_{r(e)} S_e = S_e P_{s(e)},$$

which will prove to be invaluable throughout the rest of this report.

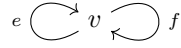
One other result that is true of all CK E -families is that $P_v \mathcal{H} = \oplus_{\{e \in E^1 \mid r(e)=v\}} S_e \mathcal{H}$. Indeed, since $P_v = \sum_{\{e \in E^1 \mid r(e)=v\}} S_e S_e^*$ is a projection then the summands must be pairwise orthogonal. That is, $S_e S_e^* S_f S_f^* = 0$ for $e \neq f$ which satisfy $r(e) = v = r(f)$. Since $S_e S_e^*$ is the projection onto the range of S_e , we have that the distinct partial isometries S_e , with range in $P_v \mathcal{H}$, have mutually orthogonal ranges. The span is clearly $P_v \mathcal{H}$ and the map which sends a sequence (h_e) in $\oplus S_e \mathcal{H}$ to $\sum_e h_e$ in $P_v \mathcal{H}$ is an isomorphism from the direct sum of the $S_e \mathcal{H}$ with $r(e) = v$ onto the space $P_v \mathcal{H}$. Thus, $P_v \mathcal{H} = \oplus_{\{e \in E^1 \mid r(e)=v\}} S_e \mathcal{H}$.

A natural question is whether a CK E -family exists for every countable graph E under the constraint that every P_v and S_e is non-zero. Indeed, we may construct such a family as follows. Choose

\mathcal{H}_v to be some separable, infinite-dimensional Hilbert space for each vertex v . We set $\mathcal{H} = \oplus_{v \in E^0} \mathcal{H}_v$ and let P_v be the projection of \mathcal{H} onto \mathcal{H}_v . We can decompose the space \mathcal{H}_v into the direct sum $\mathcal{H}_v = \oplus_{r(e)=v} \mathcal{H}_{v,e}$ where each $\mathcal{H}_{v,e}$ is again infinite-dimensional and take S_e to be the unitary isomorphism of $\mathcal{H}_{s(e)}$ onto $\mathcal{H}_{r(e),e}$ viewed as a partial isometry on \mathcal{H} with initial space $\mathcal{H}_{s(e)}$ and final space $\mathcal{H}_{r(e),e}$.

Definition 2.4. A **degenerate** CK E -family is one where some orthogonal projection P_v is the zero operator, and we concern ourselves with **non-degenerate** CK E -families.

Example 2.5. We will see this in action to get a grasp on how it works. Consider Graph 2.



GRAPH 2

The CK-conditions are $S_e^* S_e = P_v$, $S_f^* S_f = P_v$, and $P_v = S_e S_e^* + S_f S_f^*$. Let $\mathcal{H}_v = \mathcal{H} = \ell^2(\mathbb{N})$ and decompose this as $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_f$. One way to decompose \mathcal{H} is to take the standard orthonormal basis $\{e_n\}$ and let $\mathcal{H}_e = \overline{\text{span}}\{e_{2n} \mid n \geq 1\}$ and $\mathcal{H}_f = \overline{\text{span}}\{e_{2n-1} \mid n \geq 1\}$. Since $\mathcal{H}_v = \mathcal{H}$, we have that P_v is simply the identity operator. Then the isometries described above will be those determined by $S_e(e_n) = e_{2n}$ and $S_f(e_n) = e_{2n-1}$ for $n \geq 1$.

This is indeed a CK E -family. By letting E_{ij} be the elementary matrix unit relative to the basis $\{e_n\}$ with a one in the (i, j) entry and zeros elsewhere, we can view S_e and S_f as the matrices $\sum_{n=1}^{\infty} E_{2n,n}$ and $\sum_{n=1}^{\infty} E_{2n-1,n}$, respectively. It is not hard to verify that $S_e^* S_e = I_{\mathcal{H}} = S_f^* S_f$ and that $S_e S_e^* = \sum_{n=1}^{\infty} E_{2n,2n}$ while $S_f S_f^* = \sum_{n=1}^{\infty} E_{2n-1,2n-1}$. Since $P_v = I_{\mathcal{H}} = S_e S_e^* + S_f S_f^*$, the collection $\{S, P\}$ is a CK E -family for this graph.

The Cuntz algebra \mathcal{O}_n is the universal C^* -algebra generated by a set $\{S_i\}_{i=1}^n$ of isometries acting on \mathcal{H} satisfying $\sum_{i=1}^n S_i S_i^* = I$ and $S_i^* S_j = \delta_{ij} I$. Moreover, it is known that \mathcal{O}_n is isomorphic to the C^* -algebra generated by any n isometries $\{S_i\}_{i=1}^n$ satisfying the above relations (see [3]). Since S_e and S_f are isometries and the CK-conditions are precisely the relations which generate \mathcal{O}_2 , we see that in the above example, $C^*(S, P) = C^*(S)$ is equal to the Cuntz algebra, \mathcal{O}_2 .

2.2. A brief investigation of dimension. The construction of \mathcal{O}_2 for Graph 2 yielded an infinite-dimensional space $P_v \mathcal{H} = \mathcal{H}$. Moreover, we can verify that any non-degenerate CK E -family will require \mathcal{H} to be infinite-dimensional. Because S_e is an isometry from $P_v \mathcal{H}$ onto $S_e \mathcal{H}$, we have that $\dim P_v \mathcal{H} = \dim S_e \mathcal{H}$. Similarly, $\dim P_v \mathcal{H} = \dim S_f \mathcal{H}$ so the fact that $P_v \mathcal{H} = S_e \mathcal{H} \oplus S_f \mathcal{H}$ implies that $\dim P_v \mathcal{H} = \dim S_e \mathcal{H} + \dim S_f \mathcal{H} = 2 \dim P_v \mathcal{H}$. Thus, $\dim P_v \mathcal{H}$ can only be 0 or ∞ . If P_v is non-zero, then the dimension must be infinite.

We further note that in general, if E is a directed graph, each projection P_v for $v \in E^0$ is non-zero, and there is a loop e at v where v is also the range of some other edge, then $\dim(P_v \mathcal{H}) = \infty$. Indeed, we have shown $P_v \mathcal{H} = \oplus_{\{f \mid r(f)=v\}} S_f \mathcal{H}$ which implies $\dim(P_v \mathcal{H}) = \sum_{\{f \mid r(f)=v\}} \dim(S_f \mathcal{H})$.

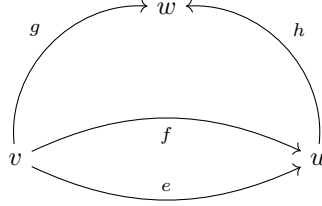
However, we know that for any edge f , $\dim(S_f \mathcal{H}) = \dim(P_{s(f)} \mathcal{H})$ and so

$$\begin{aligned} \dim(P_v \mathcal{H}) &= \dim(P_v \mathcal{H}) + \sum_{\{f \neq e \mid r(f)=v\}} \dim(S_f \mathcal{H}) \\ &= \dim(P_v \mathcal{H}) + \sum_{\{f \neq e \mid r(f)=v\}} \dim(P_{s(f)} \mathcal{H}). \end{aligned}$$

Since we assume $P_v \neq 0$ for all $v \in E^0$, $\dim(P_{s(f)} \mathcal{H})$ is nonzero and so $\dim(P_v \mathcal{H}) = \infty$.

The next natural question may be whether there exist graphs E for which it is possible to construct a CK E -family acting on a finite-dimensional Hilbert space \mathcal{H} . The following example will show that it is indeed possible.

Example 2.6. Let's consider the graph E given in Graph 3.



GRAPH 3

We start by looking at the constraints on the dimension of the subspaces $P_v \mathcal{H}$, $P_w \mathcal{H}$ and $P_u \mathcal{H}$, and seeing that

$$\begin{aligned} \dim(P_u \mathcal{H}) &= \dim(S_e \mathcal{H}) + \dim(S_f \mathcal{H}) \\ &= 2 \dim(P_v \mathcal{H}) \end{aligned}$$

$$\begin{aligned} \dim(P_w \mathcal{H}) &= \dim(S_h \mathcal{H}) + \dim(S_g \mathcal{H}) \\ &= \dim(P_u \mathcal{H}) + \dim(P_v \mathcal{H}) \\ &= 3 \dim(P_v \mathcal{H}). \end{aligned}$$

If one of $P_v \mathcal{H}$, $P_w \mathcal{H}$ or $P_u \mathcal{H}$ is infinite-dimensional, they are forced to all be infinite-dimensional. Let's try to construct an appropriate CK E -family such that $\dim(P_v \mathcal{H}) = 1$, $\dim(P_u \mathcal{H}) = 2$, and $\dim(P_w \mathcal{H}) = 3$. Since we require P_v , P_w and P_u to be mutually orthogonal, the smallest dimension of \mathcal{H} that we can consider is $3 + 2 + 1 = 6$. We will attempt to construct such a system in $M_6(\mathbb{C})$. Let's take $\{v\}$ to be a basis of $P_v \mathcal{H}$, $\{u_1, u_2\}$ a basis of $P_u \mathcal{H}$ and $\{w_1, w_2, w_3\}$ a basis of $P_w \mathcal{H}$ and list the basis elements of \mathcal{H} in the order

$$v \quad u_1 \quad u_2 \quad w_1 \quad w_2 \quad w_3.$$

Let S_e send v to u_1 and S_f send v to u_2 so that we have $S_e = E_{21}$ and $S_f = E_{31}$. Next, let's have S_g send v to w_1 so $S_g = E_{41}$. Now, set S_h to send $P_u\mathcal{H}$ to $P_w\mathcal{H}$ so we send u_1 to w_2 and u_2 to w_3 . This gives us the partial isometry $S_h = E_{52} + E_{63}$.

The CK-conditions tell us that we are forced to set $P_v = E_{11}$, $P_u = E_{22} + E_{33}$ and $P_w = E_{44} + E_{55} + E_{66}$.

We can now check the remaining CK-conditions to see that

$$\begin{aligned} S_e^* S_e &= E_{12} E_{21} = E_{11} = P_v; \\ S_f^* S_f &= E_{13} E_{31} = E_{11} = P_v; \\ S_g^* S_g &= E_{14} E_{41} = E_{11} = P_v; \\ S_h^* S_h &= (E_{25} + E_{36})(E_{52} + E_{63}) = E_{22} + E_{33} = P_u; \\ S_e S_e^* + S_f S_f^* &= E_{21} E_{12} + E_{31} E_{13} = E_{22} + E_{33} = P_u; \\ S_h S_h^* + S_g S_g^* &= (E_{52} + E_{63})(E_{25} + E_{36}) + E_{41} E_{14} = E_{44} + E_{55} + E_{66} = P_w, \end{aligned}$$

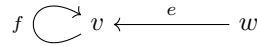
and so $\{S, P\}$ is indeed a CK E -family.

Let's now consider the C^* -algebra generated by the the CK E -family, $C^*(S, P)$. Note that we have $E_{11}, E_{21}, E_{31}, E_{41}$ all contained in the C^* -algebra. Since we can write $E_{15} = E_{12}(E_{25} + E_{36}) = S_e^* S_h^*$ and $E_{16} = E_{13}(E_{25} + E_{36}) = S_f^* S_h^*$ then each of these are also contained in $C^*(S, P)$. Now any matrix unit E_{ij} is equal to $E_{i1} E_{j1}^*$ forcing each matrix unit to be contained in $C^*(S, P)$. Thus, $C^*(S, P)$ must be all of $M_6(\mathbb{C})$.

It was argued above that we are able to represent $C^*(S, P)$ on an infinite-dimensional Hilbert space \mathcal{H} . As noted earlier, this tells us that all of \mathcal{H}_v , \mathcal{H}_u and \mathcal{H}_w will also be infinite-dimensional.

In order to completely exhaust this investigation of dimension restrictions, we ask whether the restriction that one space, \mathcal{H}_v , is infinite-dimensional gives some further restrictions on the dimension of the other spaces, \mathcal{H}_w .

Example 2.7. Let's consider the simplest graph E where this comes up, given in Graph 4.



GRAPH 4

Since there is a loop f at the vertex v and since v is also in the range of another edge e , then the remark above forces $\dim(P_v\mathcal{H})$ to be infinite-dimensional. The question now is whether there are any hidden restrictions on the dimension of $\mathcal{H}_w = P_w\mathcal{H}$?

We begin by searching for an explicit CK E -family such that $\dim(P_w\mathcal{H}) = \dim(\mathcal{H}_w) = \infty$. Take $\{S, P\}$ in $\mathcal{B}(\ell^2(\mathbb{N}))$ defined by

$$S_e = \sum_{n=1}^{\infty} E_{4n-1, 2n-1} \quad S_f = \sum_{n=1}^{\infty} E_{4n-3, 2n}$$

$$P_v = S_e^* S_e = \text{SOT} - \sum_{n=1}^{\infty} E_{2n-1, 2n-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

$$P_w = S_f^* S_f = \text{SOT} - \sum_{n=1}^{\infty} E_{2n, 2n} = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & & \ddots \end{bmatrix}$$

Then

$$S_e S_e^* + S_f S_f^* = \text{SOT} - \sum_{n=1}^{\infty} E_{4n-1, 4n-1} + \text{SOT} - \sum_{n=1}^{\infty} E_{4n-3, 4n-3} = \text{SOT} - \sum_{n=1}^{\infty} E_{2n-1, 2n-1} = P_v.$$

Thus, $\{S, P\}$ is a CK E -family and in this case, $\dim(P_w \mathcal{H}) = \infty$. We now attempt to find a CK E -family $\{T, Q\}$ such that $\dim(\mathcal{H}_w) < \infty$. Define $\{T, Q\}$ by

$$T_e = E_{21} \quad T_f = \text{SOT} - \sum_{n=1}^{\infty} E_{n+2, n+1},$$

so that the projections must be set to be

$$Q_v = T_f^* T_f = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \quad Q_w = T_e^* T_e = E_{11}$$

where the bold face $\mathbf{0}$ is an infinite matrix of zeros. Finally, $T_e T_e^* + T_f T_f^* = E_{22} + \sum_{n=3}^{\infty} E_{nn} = Q_v$ as desired, proving that $\{T, Q\}$ is a CK family for the graph above with $\dim(P_w \mathcal{H}) = 1 < \infty$. We will determine what $C^*(S, P)$ and $C^*(T, Q)$ are later in this report.

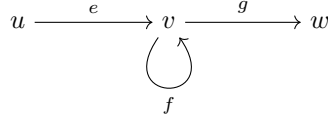
2.3. Paths and a few results. We now expand our theory to the case of paths.

Definition 2.8. We define a **path** to be a sequence $\mu = \mu_1 \mu_2 \dots$ of edges μ_i in E^1 such that $s(\mu_i) = r(\mu_{i+1})$. We let $|\mu|$ denote the number of edges in μ and let E^n be the set of paths of length n .

Our previous notation of E^1 being the set of edges and E^0 the set of vertices (considered to be paths of length 0) is consistent with this new notation. Finally, we let E^* be the set of all paths of finite length. We extend the functions r and s in the obvious way: $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$, where we may extend r to all paths and s to all paths of finite length.

We give a simple example to demonstrate the notation, as it can vary in the literature.

Example 2.9. We now look at Graph 5.



GRAPH 5

Consider the path μ of length 3 defined to be $\mu = gfe$ and note that the vertex v is repeated in μ . We may also repeat the edge f and instead consider the path $\nu = gfffe$ of length 4. Both of these paths have source at u and range at w .

Much of the theory will be similarly expanded for paths. For the sequence $\mu = \mu_1\mu_2\ldots\mu_n$, we define $S_\mu = S_{\mu_1}S_{\mu_2}\ldots S_{\mu_n}$, where we convene that $S_v = P_v$ when considering paths of length 0. If we take two edges e and f with $s(e) \neq r(f)$ (that is, ef is not a path) then $S_{ef} = S_eS_f = S_eP_{s(e)}P_{r(f)}S_f = 0$ where the last equality holds true because the projections $P_{s(e)}$ and $P_{r(f)}$ are mutually orthogonal. Thus, if $P_v \neq 0$ for all $v \in E^0$, then S_μ is non-zero if and only if μ is a path in our graph.

For a path $\mu \in E^*$, we have $S_\mu^*S_\mu = P_{s(\mu)}$, so the S_μ 's are also partial isometries. Moreover, $S_\mu = P_{r(\mu)}S_\mu = S_\mu P_{s(\mu)}$.

Proposition 2.10. *Let E be a row-finite graph and $\{S, P\}$ a CK E -family in a C^* -algebra \mathfrak{B} . Then for every vertex v and $k \in \mathbb{Z}_{\geq 0}$,*

$$P_v = \sum_{\substack{r(\mu)=v \\ |\mu|=k}} S_\mu S_\mu^* + \sum_{\substack{r(\mu)=v \\ |\mu|<k \text{ and} \\ s(\mu) \text{ is a source}}} S_\mu S_\mu^*.$$

Proof. We will prove this by induction on k . If $k = 0$ then the equation holds trivially. Now suppose the equation holds for some non-negative number k , and we will show it also holds for $k + 1$.

Firstly, note that because E is row-finite, there are at most finitely many paths $\mu \in E^n$ with $r(\mu) = v$. We may now manipulate the induction hypothesis for k :

$$\begin{aligned}
P_v &= \sum_{\substack{r(\mu)=v \\ |\mu|=k}} S_\mu S_\mu^* + \sum_{\substack{r(\mu)=v \\ |\mu|<k \text{ and} \\ s(\mu) \text{ is a source}}} S_\mu S_\mu^* \\
&= \sum_{\substack{r(\mu)=v \\ |\mu|=k \text{ and} \\ s(\mu) \text{ is not a source}}} S_\mu P_{s(\mu)} S_\mu^* + \sum_{\substack{r(\mu)=v \\ |\mu|\leq k \text{ and} \\ s(\mu) \text{ is a source}}} S_\mu S_\mu^*,
\end{aligned}$$

so that

$$\begin{aligned}
P_v &= \sum_{\substack{r(\mu)=v \\ |\mu|=k \text{ and} \\ s(\mu) \text{ is not a source}}} S_\mu \left(\sum_{r(e)=s(\mu)} S_e S_e^* \right) S_\mu^* + \sum_{\substack{r(\mu)=v \\ |\mu|\leq k \text{ and} \\ s(\mu) \text{ is a source}}} S_\mu S_\mu^* \\
&= \sum_{\substack{r(\mu)=v \\ |\mu|=k \text{ and} \\ s(\mu) \text{ is not a source}}} \sum_{r(e)=s(\mu)} S_{\mu e} S_{\mu e}^* + \sum_{\substack{r(\mu)=v \\ |\mu|\leq k \text{ and} \\ s(\mu) \text{ is a source}}} S_\mu S_\mu^*.
\end{aligned}$$

Thus, the equation holds for $k+1$, since $|\mu e| = k+1$. □

The following proposition will be used multiple times throughout the rest of this report:

Proposition 2.11. *Let E be a row-finite graph and suppose $\{S, P\}$ is a CK E -family in a C^* -algebra \mathfrak{B} and that $\mu = \mu_1 \mu_2 \dots \mu_n$ and $\nu = \nu_1 \nu_2 \dots \nu_m$ are paths in E . Then*

$$S_\mu^* S_\nu = \begin{cases} S_{\mu'}^* & \text{if } \mu = \nu \mu' \text{ for some } \mu' \in E^* \\ S_{\nu'} & \text{if } \nu = \mu \nu' \text{ for some } \nu' \in E^* \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose first that $|\mu| = |\nu| = n$ and let i be the smallest integer such that $\mu_i \neq \nu_i$. Then

$$\begin{aligned}
S_\mu^* S_\nu &= S_{\mu_n}^* \dots S_{\mu_i}^* (S_{\mu_{i-1}}^* \dots S_{\mu_1}^*) (S_{\mu_1} S_{\mu_2} \dots S_{\mu_{i-1}}) S_{\nu_1} \dots S_{\nu_n} \\
&= S_{\mu_n}^* \dots S_{\mu_i}^* P_{r(\mu_i)} S_{\nu_i} \dots S_{\nu_n} && \text{since } s(\mu_{i-1}) = r(\mu_i) \\
&= S_{\mu_n}^* \dots S_{\mu_i}^* S_{\nu_i} \dots S_{\nu_n}.
\end{aligned}$$

Now, $S_{\mu_i}^* S_{\nu_i} = S_{\mu_i} (S_{\mu_i} S_{\mu_i}^*) (S_{\nu_i} S_{\nu_i}^*) S_{\nu_i} = 0$. Next, let us assume that $n = |\mu| < |\nu|$ and write $\nu = \alpha \nu'$ with $|\alpha| = n$ so that $S_\mu^* S_\nu = (S_\mu^* S_\alpha) S_{\nu'}$. If $\mu = \alpha$ then $S_\mu^* S_\nu = P_{s(\mu)} S_{\nu'} = P_{r(\nu')} S_{\nu'} = S_{\nu'}$.

If $\mu \neq \alpha$ then $S_\mu^* S_\nu = (S_\mu^* S_\alpha) S_\nu = 0$. A similar argument for the case of $|\mu| > |\nu|$ will conclude the proof. \square

We will now begin looking at the C^* -algebra generated by a CK E -family.

Theorem 2.12. *If $\{S, P\}$ is a CK E -family for a row-finite graph E , then*

$$C^*(S, P) = \overline{\text{span}}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

Proof. First note that every non-zero finite product of the partial isometries S_e and S_f^* has the form $S_\mu S_\nu^*$ for some $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$.

More specifically, we have

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases}$$

This tells us that $\text{span}\{S_\mu S_\nu^* \mid \mu, \nu \in E^*, s(\mu) = s(\nu)\}$ is a subalgebra of $C^*(S, P)$. Moreover, it is a $*$ -subalgebra since $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$. Thus, the closure is a C^* -subalgebra of $C^*(S, P)$ and so we've shown one inclusion.

The other inclusion follows from the fact that $S_e S_{s(e)}^* = S_e P_{s(e)}^* = S_e P_{s(e)} = S_e$ and $S_v S_v^* = P_v$, so the generators are in the span. Thus, the two spaces are equal. \square

Definition 2.13. For finite paths α and ν , we say that α **extends** ν if there exists some path α' which satisfies $\alpha = \nu\alpha'$. We define a **closed path** to be a path μ which satisfies $r(\mu) = s(\mu)$ and we define a **cycle** to be a closed path μ of length at least one which does not repeat vertices. We call E a **finite graph** if it has finitely many vertices and edges.

We now have the necessary terminology to present our next theorem.

Theorem 2.14. *Suppose E is a finite graph without cycles. Let w_1, \dots, w_n be the sources in E . Then for every Cuntz-Krieger E -family $\{S, P\}$ with non-zero projections P_v we have*

$$C^*(S, P) \cong \oplus_{i=1}^n M_{|s^{-1}(w_i)|}(\mathbb{C})$$

where $s^{-1}(w_i) = \{\mu \in E^* \mid s(\mu) = w_i\}$.

Proof. First, let's confirm that the graph E has sources. Choose any vertex v in E^0 . If v is a source, we've shown the claim. If not, it's possible to find an edge e with $r(e) = v$. If $s(e)$ is a source then the claim is again verified. If not, we can find another edge into $s(e)$. Because there are finitely vertices, repeating the argument above will eventually lead to either repeating a vertex (thus creating a cycle, leading to a contradiction) or terminating at a source.

Note that for any two paths $\mu, \nu \in E^*$ with $s(\mu) = s(\nu) = v$, where v is not one of the sources, we can write

$$S_\mu S_\nu^* = S_\mu P_v S_\nu^* = S_\mu \left(\sum_{r(e)=v} S_e S_e^* \right) S_\nu^* = \sum_{r(e)=v} S_{\mu e} S_{\nu e}^*.$$

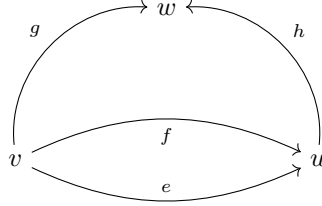
We have now extended the length of the paths μ and ν . We can continue to use the CK-relation to extend the paths until $s(\mu) = s(\nu)$ is some source, w_i . Theorem 2.12 combined with this argument tells us that

$$C^*(S, P) = \text{span}\{S_\mu S_\nu^* \mid s(\mu) = s(\nu) = w_i \text{ for some } i\}.$$

For two paths $\nu, \alpha \in E^*$ with $s(\nu) = s(\alpha) = w_i$ for some fixed source w_i , we have that ν cannot extend α and neither can α extend ν . Thus, if $\delta_{\mu, \nu}$ is the Kronecker product, $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \delta_{\nu, \alpha} S_\mu S_\beta^*$ making these $S_\mu S_\nu^*$ matrix units. Since there are $|s^{-1}(w_i)|^2$ such matrix units, $\text{span}\{S_\mu S_\nu^* \mid s(\mu) = s(\nu) = w_i\}$ is isomorphic to $M_{|s^{-1}(w_i)|}(\mathbb{C})$.

Finally, if we have the element $S_\mu S_\nu^*$ with $s(\mu) = s(\nu) = w_i$ as well as $S_\alpha S_\beta^*$ with $s(\alpha) = s(\beta) = w_j$ for $i \neq j$ then ν and α cannot extend each other and thus $(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = 0$, giving us the desired direct sum of $C^*(S, P) = \oplus_{i=1}^n \text{span}\{S_\mu S_\nu^* \mid s(\mu) = s(\nu) = w_i\}$. \square

Example 2.15. Recall Graph 3 we examined earlier which had no cycles, shown again below.



In Section 2.2, we defined a particular CK E -family $\{S, P\}$ and found that $C^*(S, P)$ is equal to $M_6(\mathbb{C})$. Because v is the only source and $s^{-1}(v) = \{v, e, f, g, hf, he\}$, the theorem above tells us that for any CK E -family $\{T, Q\}$ we have $C^*(T, Q) \cong M_6(\mathbb{C})$.

To see the proof of the theorem above in action for this example, we show why P_u and S_h are contained in $\text{span}\{S_\mu S_\nu^* \mid s(\mu) = s(\nu) = v\}$:

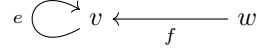
$$P_u = S_e S_e^*$$

$$\begin{aligned} S_h &= S_h P_u \\ &= S_h (S_f S_f^* + S_e S_e^*) \\ &= S_{hf} S_f^* + S_{he} S_e^* \end{aligned}$$

Similarly, we can write any element of $C^*(S, P)$ as $S_\mu S_\nu^*$ for an appropriate choice of paths $\mu, \nu \in s^{-1}(v)$.

We are now able to characterize $C^*(S, P)$ for any graph E with no cycles. What if the graph does have a cycle?

Example 2.16. Let's consider Graph 4 with cycles that we looked at earlier.



We have the Cuntz-Krieger conditions $S_e^* S_e = P_v$, $S_f^* S_f = P_w$, and $P_v = S_e S_e^* + S_f S_f^*$.

Then by applying the previous remarks, we have that

$$\begin{aligned} (P_v + P_w)P_v &= P_v = P_v(P_v + P_w), \\ (P_v + P_w)P_w &= P_w = P_w(P_v + P_w), \\ (P_v + P_w)S_e &= P_v S_e + P_w(P_v S_e) = S_e = S_e(P_v + P_w), \text{ and} \\ (P_v + P_w)S_f &= P_v(P_w S_f) + P_w S_f = S_f = S_f(P_v + P_w), \end{aligned}$$

proving that $P_v + P_w$ is the identity for $C^*(S, P)$. Moreover,

$$\begin{aligned} (S_e + S_f)^*(S_e + S_f) &= S_e^* S_e + S_f^* S_e + S_e^* S_f + S_f^* S_f = P_v + P_w; \\ (S_e + S_f)(S_e + S_f)^* &= S_e S_e^* + S_f S_e^* + S_e S_f^* + S_f S_f^* = P_v; \\ (S_e + S_f)^*(S_e + S_f) - P_v &= P_w; \\ (S_e + S_f)P_v &= S_e P_v + S_f(S_e S_e^* + S_f S_f^*) = S_e + S_f S_e S_e^* + S_f S_f S_f^* = S_e; \\ (S_e + S_f)P_w &= S_e(S_f^* S_f) + S_f P_w = S_f. \end{aligned}$$

This argument shows that we can recover the generators of $C^*(S, P)$ from $S_e + S_f$. Thus, we may recover all of $C^*(S, P)$ from $S_e + S_f$. This precisely means that $C^*(S, P)$ is generated by the isometry $S_e + S_f$.

Conversely, if V is an isometry then $P_w = I - VV^*$, $P_v = VV^*$, $S_e = VP_v$, $S_f = VP_w$ defines a CK E -family (this is easy to check) and $C^*(S, P) = C^*(V)$. Coburn's Theorem [8, Theorem 3.5.18.] tells us that all C^* -algebras generated by a single non-unitary isometry are isomorphic to the Toeplitz algebra \mathcal{T} .

Since $S_e + S_f$ is non-unitary precisely when P_w is non-zero then we know that all CK E -families with non-zero projections generate C^* -algebras which are isomorphic to \mathcal{T} .

2.4. The universal C^* -algebra of a graph. In the previous example, we were required to use Coburn's Theorem to prove that the C^* -algebra generated by any CK E -family was isomorphic to \mathcal{T} . This cannot be extended to apply to an arbitrary graph. Instead, we will present a theorem which defines a universal C^* -algebra, $C^*(E)$. This will be called the C^* -algebra of the graph E .

Theorem 2.17. *For a given row-finite graph E there is a C^* -algebra, $C^*(E)$, generated by a CK E -family $\{s, p\}$, such that for every C^* -algebra \mathfrak{B} and for every CK E -family $\{T, Q\}$ in \mathfrak{B} , there exists a homomorphism*

$$\pi_{T,Q} : C^*(E) \rightarrow \mathfrak{B}$$

which maps s_e to T_e and p_v to Q_v . Moreover, this construction is unique. That is, suppose \mathfrak{C} is a C^ -algebra generated by a CK-family $\{w, r\}$ such that, for every CK-family $\{T, Q\}$ in \mathfrak{B} , there is a homomorphism*

$$\rho_{T,Q} : \mathfrak{C} \rightarrow \mathfrak{B}$$

which maps w_e to T_e and r_v to Q_v . Under these conditions, there exists an isomorphism $\varphi : C^(E) \rightarrow \mathfrak{C}$ such that $\varphi(s_e) = w_e$ and $\varphi(p_v) = r_v$.*

Proof. We take formal symbols $d_{\mu\nu}$ for paths $\mu, \nu \in E^*$ and consider the set

$$V = \left\{ \sum z_{\mu\nu} d_{\mu\nu} \mid \mu, \nu \in E^* \quad s(\mu) = s(\nu) \right\},$$

equipped with the operations

$$a \left(\sum w_{\mu\nu} d_{\mu\nu} \right) + \left(\sum z_{\mu\nu} d_{\mu\nu} \right) = \sum (aw_{\mu\nu} + z_{\mu\nu}) d_{\mu\nu};$$

$$d_{\mu\nu}^* = d_{\nu\mu}; \text{ and}$$

$$d_{\mu\nu} d_{\alpha\beta} = \begin{cases} d_{\mu\alpha',\beta} & \text{if } \alpha = \nu\alpha' \\ d_{\mu,\beta\nu'} & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise} \end{cases}.$$

We can check that product is associative and compatible with $*$, making V a $*$ -algebra. For any CK E -family $\{S, P\}$ generating the C^* -algebra \mathfrak{A} , the map

$$\begin{aligned} \pi_{S,P} : V &\rightarrow \mathcal{B}(\mathcal{H}) \\ d_{\mu\nu} &\mapsto s_\mu s_\nu^* \end{aligned}$$

is a $*$ -homomorphism since $\{s_\mu s_\nu^*\}$ satisfy the relations above. Moreover,

$$\left\| \pi_{S,P} \left(\sum z_{\mu\nu} d_{\mu\nu} \right) \right\| \leq \sum |z_{\mu\nu}| \|s_\mu s_\nu^*\| \leq \sum |z_{\mu\nu}|,$$

so for all $a \in V$, we may define

$$\|a\|_1 = \sup\{\|\pi_{s,p}(a)\| \mid \{s, p\} \text{ is a CK family}\} < \infty.$$

This is a seminorm on V and $\|a\|_1^2 = \|a^*a\|_1$. Consider the space $I = \{a \in V \mid \|a\|_1 = 0\}$, which is a $*$ -ideal. Let $V_0 = V/I$ be the $*$ -algebra with quotient norm $\|\cdot\|_0$ defined by $\|v + I\|_0 = \inf\{\|v + j\|_1 \mid j \in I\}$.

It follows that $\overline{V_0}$, the closure of V_0 with respect to $\|\cdot\|_0$, is a C^* -algebra, so we may let $C^*(E) = \overline{V_0}$. Consider $s_e = d_{e,s(e)}, p_v = d_{v,v}$. We claim that this is a CK E -family which generates V_0 . Indeed,

$$p_v^2 = d_{v,v}d_{v,v} = d_{v,v} = p_v = d_{v,v}^* = p_v^*$$

$$p_v p_w = d_{v,v}d_{w,w} = 0 \text{ if } v \neq w$$

$$s_e^* s_e = \sum_{r(e)=v} d_{e,s(e)} d_{e,s(e)}^* = \sum_{r(e)=v} d_{e,s(e)} d_{s(e),e} = \sum_{r(e)=v} d_{e,e}.$$

We now show that $\sum_{r(e)=v} d_{e,e} = d_{v,v}$. Indeed,

$$\begin{aligned} \left\| \sum_{r(e)=v} d_{e,e} - d_{v,v} \right\|_0 &= \inf \left\{ \left\| \sum_{r(e)=v} d_{e,e} - d_{v,v} + j \right\|_1 \mid j \in I \right\} \\ &\leq \left\| \sum_{r(e)=v} d_{e,e} - d_{v,v} \right\|_1 \\ &= \sup \left\{ \left\| \pi_{S,P} \left(\sum_{r(e)=v} d_{e,e} - d_{v,v} \right) \right\| \mid \{s, p\} \text{ is a CK family} \right\} \\ &= \sup \left\{ \left\| \sum_{r(e)=v} s_e s_e^* - p_v \right\| \mid \{s, p\} \text{ is a CK family} \right\} \\ &= 0. \end{aligned}$$

To see $\{s, p\}$ generates V_0 , we note that if e, f are two edges with $s(e) = s(f)$ then $d_{e,s(e)} d_{f,s(f)}^* = d_{e,s(e)} d_{s(f),f} = d_{e,f}$. If ef is a path, then $d_{e,s(e)} d_{f,s(f)} = d_{ef,s(f)} = d_{ef,s(ef)}$. Indeed, $\{s, p\}$ generates V_0 . Now, we can consider a faithful representation $\rho : \mathfrak{B} \rightarrow \mathcal{B}(\mathcal{H})$. Set $\pi_{T,Q} = \rho^{-1} \circ \pi_{\rho(T),\rho(Q)} : V_0 \rightarrow \mathfrak{B}$ so that

$$\begin{aligned}
\pi_{T,Q}(s_e) &= \pi_{T,Q}(d_{e,s(e)}) \\
&= \rho^{-1}(\pi_{\rho(T),\rho(Q)}(d_{e,s(e)})) \\
&= \rho^{-1}(\rho(T_e)\rho(Q_{s(e)})) \\
&= \rho^{-1}(\rho(T_e Q_{s(e)})) \\
&= \rho^{-1}(\rho(T_e)) \\
&= T_e, \text{ and}
\end{aligned}$$

$$\begin{aligned}
\pi_{T,Q}(p_v) &= \rho^{-1}(\pi_{\rho(T),\rho(Q)}(d_{v,v})) \\
&= \rho^{-1}(\rho(Q_v)\rho(Q_v)) \\
&= \rho^{-1}(\rho(Q_v)) \\
&= Q_v.
\end{aligned}$$

Thus, $\pi_{T,Q}$ does the trick. We are only left to prove the universal property. Since $\{w, r\}$ is a CK E -family, there exists a map $\pi_{w,r} : C^*(E) \rightarrow \mathcal{C}$. We are left to prove that this is an isomorphism.

Note that $w_e = w_e w_{s(e)}^* = \pi_{w,r}(d_{e,s(e)})$ and $r_v = w_v w_v^* = \pi_{w,r}(d_{v,v})$, so since $\text{ran}(\pi_{w,r})$ contains the generators of \mathcal{C} , it is surjective.

Conversely, there exists a $\rho_{s,p} : \mathcal{C} \rightarrow C^*(E)$ so that

$$\begin{aligned}
(\rho_{s,p} \circ \pi_{w,r})(s_e) &= \rho_{s,p}(w_e) = s_e \\
(\rho_{s,p} \circ \pi_{w,r})(p_v) &= \rho_{s,p}(r_v) = p_v.
\end{aligned}$$

So $\rho_{s,p} \circ \pi_{w,r}$ is the identity on $C^*(E)$. Thus, $\pi_{w,r}(a) = 0$ implies $\rho_{s,p}(\pi_{w,r}(a)) = 0$ so that $a = 0$ and $\pi_{w,r}$ is injective, which finally forces $\mathcal{C} \cong C^*(E)$. \square

For the rest of this report, we will use lower case letters for a CK E -family only when we suppose it has the universal property described above.

Example 2.18. Let's apply this theorem to find the C^* -algebra for Graph 6.



GRAPH 6

We have the Cuntz-Krieger conditions $s_e^* s_e = p_v = s_e s_e^*$, where $C^*(E)$ is generated by $\{s, p\}$. Since $C^*(E)$ is generated by s_e , then p_v is the identity on $C^*(E)$ making s_e a unitary operator.

We consider the C^* -algebra $\mathcal{C}(\mathbb{T})$ where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. Consider the two functions

$$\begin{aligned} t_e = \iota : z &\mapsto z && \text{(inclusion function)} \\ q_v = \mathbf{1} : z &\mapsto 1. && \text{(constant)} \end{aligned}$$

Then $\{t, q\}$ is a CK E -family and by the Stone-Weierstrass Theorem we have $C^*(t, q) = \mathcal{C}(\mathbb{T})$. By the universal property of $C^*(E)$, there is a $*$ -homomorphism

$$\begin{aligned} \varphi : C^*(E) &\rightarrow \mathcal{C}(\mathbb{T}) \\ s_e &\mapsto t_e = \iota \\ p_v &\mapsto q_v = \mathbf{1}. \end{aligned}$$

Clearly, φ is surjective. From the continuous functional calculus, we have the map

$$\begin{aligned} \Psi : \mathcal{C}(\sigma(s_e)) &\rightarrow C^*(s_e) = C^*(E) \\ f &\mapsto f(s_e) \end{aligned}$$

where Ψ maps t_e to s_e and q_v to p_v . Now, $\mathbb{T} = \sigma(\iota) = \sigma(\varphi(s_e)) \subseteq \sigma(s_e) \subseteq \mathbb{T}$, so $\sigma(s_e) = \mathbb{T}$, which proves that Ψ is the inverse of φ . Thus, φ is an isomorphism and $C^*(E) \cong \mathcal{C}(\mathbb{T})$.

3. UNIQUENESS THEOREMS

Under certain conditions we can guarantee that each of the C^* -algebras generated by a given CK E -family will be isomorphic. In this section, we will investigate these conditions. These theorems are put into practice to find the universal C^* -algebra for certain graphs.

3.1. Gauge-invariant uniqueness. We will now work towards proving the first uniqueness theorem. This theorem will utilize the existence of a gauge action on the C^* -algebra \mathfrak{B} to prove uniqueness.

Definition 3.1. Given a locally compact group G and a C^* -algebra \mathfrak{A} , we say a map γ from G into the automorphisms $\text{Aut } \mathfrak{A}$ is **strongly continuous** if for any fixed element $a \in \mathfrak{A}$, the map $z \mapsto \gamma_z(a)$ is continuous. We define an **action** of G on \mathfrak{A} to be a homomorphism $\alpha : G \rightarrow \text{Aut } \mathfrak{A}$ which sends g to α_g and is strongly continuous.

The following result proves the existence of a particular action.

Theorem 3.2. *For any graph E , there is an action $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E))$ such that $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$.*

Proof. Take $\{s, p\}$ to be a CK E -family which generates $C^*(E)$. For a fixed z in \mathbb{T} , we have $z\bar{z} = 1$. It is then easy to verify that $\{zs, p\}$ is a CK E -family which generates $C^*(E)$. Similarly, if $\{T, Q\}$ is a CK E -family in a C^* -algebra \mathfrak{B} , so is $\{\bar{z}T, Q\}$, and using the notation of the previous theorem, we have that $\pi_{\bar{z}T, Q}(zs_e) = z\pi_{\bar{z}T, Q}(s_e) = z(\bar{z}T_e) = T_e$.

If we set $\rho_{T,Q} = \pi_{\bar{z}T,Q}$ then $C^*(E)$ generated by $\{zs, p\}$ has the universal property. Applying Theorem 2.17, we can find an isomorphism

$$\begin{aligned}\gamma_z : C^*(E) &\rightarrow C^*(E) \\ s_e &\mapsto zs_e \\ p_v &\mapsto p_v.\end{aligned}$$

For w in \mathbb{T} , we have that $\gamma_z \circ \gamma_w$ and γ_{zw} agree on the generators and so they must be equal on all of $C^*(E)$. Thus, $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(E))$ given by $\gamma(z) = \gamma_z$ is a homomorphism. To finish the proof, we require γ to be strongly continuous, that is, for any fixed element a in $C^*(E)$ we require the map $z \mapsto \gamma_z(a)$ to be continuous. We check that now. Fix $\epsilon > 0$ and choose $c = \sum \lambda_{\mu,\nu} s_\mu s_\nu^*$ such that $\|a - c\| < \epsilon/3$. Then

$$\gamma_z(s_\mu) = \gamma_z(s_{\mu_1} \dots s_{\mu_{|\mu|}}) = \gamma_z(s_{\mu_1}) \dots \gamma_z(s_{\mu_{|\mu|}}) = zs_{\mu_1} \dots zs_{\mu_{|\mu|}} = z^{|\mu|} s_{\mu_1} \dots s_{\mu_{|\mu|}} = z^{|\mu|} s_\mu,$$

and so

$$\gamma_z(s_\mu s_\nu^*) = \gamma_z(s_\mu) \gamma_z(s_\nu^*) = z^{|\mu|} s_\mu z^{-|\nu|} s_\nu^* = z^{|\mu|-|\nu|} s_\mu s_\nu^*.$$

Since scalar multiplication is continuous, so is the map $w \mapsto \gamma_w(c) = \sum \lambda_{\mu,\nu} w^{|\mu|-|\nu|} s_\mu s_\nu^*$. Therefore, for the fixed ϵ , we can find some $\delta > 0$ such that $|w - z| < \delta$ implies that $\|\gamma_w(c) - \gamma_z(c)\| < \epsilon/3$.

Since automorphisms of C^* -algebras preserve the norm, $\|\gamma_z(a - c)\| < \epsilon/3$, which gives

$$\|\gamma_w(a) - \gamma_z(a)\| \leq \|\gamma_w(a - c)\| + \|\gamma_w(c) - \gamma_z(c)\| + \|\gamma_z(a - c)\| < 3 \frac{\epsilon}{3} = \epsilon,$$

and so the map $z \mapsto \gamma_z(a)$ is continuous for each fixed $a \in C^*(E)$ and the theorem holds. \square

We call this action γ a **gauge action**. Throughout the remainder of this report, γ will denote the gauge action.

Definition 3.3. Let E be a graph and α an action of \mathbb{T} on $C^*(E)$. Define the **fixed point algebra**, $C^*(E)^\alpha$, to be

$$C^*(E)^\alpha = \{a \in C^*(E) \mid \alpha_z(a) = a \text{ for all } z \in \mathbb{T}\}.$$

Note that for any $\mu \in E^*$ we have $\gamma_z(s_\mu) = z^{|\mu|} s_\mu$ and $\gamma_z(s_\mu^*) = z^{-|\mu|} s_\mu^*$ which gives $\gamma_z(s_\mu s_\nu^*) = z^{|\mu|-|\nu|} s_\mu s_\nu^*$. Hence, the elements $s_\mu s_\nu^*$ with $|\mu| = |\nu|$ are fixed points of γ_z . Thus, $\overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\} \subseteq C^*(E)^\gamma$.

Lemma 3.4. For a continuous $f : \mathbb{T} \rightarrow \mathfrak{A}$, there exists a unique element $\int_{\mathbb{T}} f(z)dz$ in \mathfrak{A} such that for every representation $\pi : \mathfrak{A} \rightarrow \mathcal{B}(\mathcal{H})$,

$$\left\langle \pi \left(\int_{\mathbb{T}} f(z)dz \right) h, k \right\rangle = \int_{\mathbb{T}} \langle \pi(f(z))h, k \rangle dz \quad \forall h, k \in \mathcal{H}$$

and

- (1) $b(\int_{\mathbb{T}} f(z)dz) = \int_{\mathbb{T}} bf(z)dz$ for any $b \in \mathfrak{A}$;
- (2) $\|\int_{\mathbb{T}} f(z)dz\| \leq \int_{\mathbb{T}} \|f(z)\|dz$;
- (3) $\phi(\int_{\mathbb{T}} f(z)dz) = \int_{\mathbb{T}} \phi(f(z))dz$ for all homomorphisms $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$; and
- (4) for $w \in \mathbb{T}$, $\int_{\mathbb{T}} f(wz)dz = \int_{\mathbb{T}} f(z)dz$.

In order to keep this report a reasonable length, we will skip the proof of this lemma, however such an element can be constructed in the usual way using Riemann sums.

Definition 3.5. For a C^* -subalgebra \mathfrak{B} of the C^* -algebra \mathfrak{A} , we call a linear map $E : \mathfrak{A} \rightarrow \mathfrak{A}$ a **conditional expectation** of \mathfrak{A} onto \mathfrak{B} if

- (1) E is positive
- (2) E is idempotent
- (3) $\|E\| \leq 1$
- (4) $\text{ran } E = \mathfrak{B}$
- (5) $E(ba) = bE(a)$ for all $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

Note that by taking adjoints, the last condition is equivalent to requiring $E(ab) = E(a)b$ for $a \in \mathfrak{A}$, $b \in \mathfrak{B}$. We call such a map E **faithful** if $E(aa^*) = 0$ implies that $a = 0$.

We direct the reader to Lemma 5.17 below for an example of such a conditional expectation.

Proposition 3.6. If α is an action of \mathbb{T} on $C^*(E)$, define the map

$$\begin{aligned} \Phi_{\alpha} : C^*(E) &\rightarrow C^*(E) \\ a &\mapsto \int_{\mathbb{T}} \alpha_z(a)dz. \end{aligned}$$

Then Φ_{α} is a faithful conditional expectation.

Proof. The fact that Φ_{α} is positive follows from α being a $*$ -homomorphism. If we take $a \in C^*(E)$ and $w \in \mathbb{T}$ we have

$$\begin{aligned} \alpha_w(\Phi_{\alpha}(a)) &= \alpha_w \left(\int_{\mathbb{T}} \alpha_z(a)dz \right) \\ &= \int_{\mathbb{T}} \alpha_w(\alpha_z(a))dz && \text{(by part (3) of lemma 3.4)} \\ &= \int_{\mathbb{T}} \alpha_{wz}(a)dz \\ &= \int_{\mathbb{T}} \alpha_z(a)dz && \text{(by part (4) of lemma 3.4)} \\ &= \Phi_{\alpha}(a). \end{aligned}$$

So $\Phi_\alpha(a) \in C^*(E)^\alpha$. Now, if $a \in C^*(E)^\alpha$, then $\Phi_\alpha(a) = \int_{\mathbb{T}} \alpha_z(a) dz = \int_{\mathbb{T}} a dz = a$ so that $\text{ran } E = C^*(E)^\alpha$. It follows from above that $\Phi_\alpha(\Phi_\alpha(a)) = \Phi_\alpha(a)$ so Φ_α is idempotent. Finally, it is easy to verify that for $a \in C^*(E)$ and $b \in C^*(E)^\alpha$,

$$\Phi_\alpha(ab) = \int_{\mathbb{T}} \alpha_z(ab) dz = \left(\int_{\mathbb{T}} \alpha_z(a) dz \right) b = \Phi_\alpha(a)b.$$

Thus, Φ_α is a conditional expectation. We now suppose $\Phi_\alpha(a^*a) = 0$ and choose π to be a faithful representation of $C^*(E)$ onto some \mathcal{H} , the existence of which is given in [2, Theorem 7.10.]. Then for any $h \in \mathcal{H}$,

$$\begin{aligned} 0 &= \langle \pi(\Phi(a^*a))h, h \rangle \\ &= \left\langle \pi \left(\int_{\mathbb{T}} \alpha_z(a^*a) dz \right) h, h \right\rangle \\ &= \int_{\mathbb{T}} \langle \pi(\alpha_z(a^*a))h, h \rangle dz && \text{(by lemma 3.4)} \\ &= \int_{\mathbb{T}} \langle \pi(\alpha_z(a^*)\alpha_z(a))h, h \rangle dz \\ &= \int_{\mathbb{T}} \langle \pi(\alpha_z(a))^* \pi(\alpha_z(a))h, h \rangle dz \\ &= \int_{\mathbb{T}} \langle \pi(\alpha_z(a))h, \pi(\alpha_z(a))h \rangle dz \\ &= \int_{\mathbb{T}} \|\pi(\alpha_z(a))h\|^2 dz. \end{aligned}$$

Since $\|\pi(\alpha_z(a))h\|$ is a non-negative continuous function, it must be equal to zero. Thus, because $h \in \mathcal{H}$ was arbitrary, $\pi(\alpha_z(a)) = 0$ for all $z \in \mathbb{T}$. In particular, $0 = \pi(\alpha_1(a)) = \pi(a)$, and because π is faithful, we have that $a = 0$. \square

In the theorem above, we may consider the gauge action γ defined in Theorem 3.2.

Corollary 3.7. *Suppose E is a row finite graph and take γ to be the gauge action. For every finite collection $F \subseteq E^*$ and for any scalars $c_{\mu\nu}$ we have*

$$(\star) \quad \Phi_\gamma \left(\sum_{\mu, \nu \in F} c_{\mu\nu} s_\mu s_\nu^* \right) = \sum_{\mu, \nu \in F, |\mu|=|\nu|} c_{\mu\nu} s_\mu s_\nu^*$$

and

$$C^*(E)^\gamma = \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\}.$$

Proof. Fix some paths $\mu, \nu \in E^*$ with $s(\mu) = s(\nu)$. Utilizing the map Φ_γ from above, we see that if $|\mu| = |\nu|$, $s_\mu s_\nu^*$ is a fixed point of Φ_γ . If their lengths are not equal then Φ_γ sends $s_\mu s_\nu^*$ to zero since in this case $\int_{\mathbb{T}} z^{|\mu|-|\nu|} dt = 0$.

Thus, we get equation (\star):

$$\Phi_\gamma \left(\sum_{\mu, \nu \in F} c_{\mu\nu} s_\mu s_\nu^* \right) = \sum_{\mu, \nu \in F} c_{\mu\nu} \Phi(s_\mu s_\nu^*) = \sum_{|\mu|=|\nu|} c_{\mu\nu} s_\mu s_\nu^*.$$

We already know that $\overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu), |\mu| = |\nu|\} \subseteq C^*(E)^\gamma$ and since Φ_γ is continuous,

$$C^*(E)^\gamma = \Phi_\gamma(C^*(E)^\gamma) \subseteq \Phi_\gamma(C^*(E)) = \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \quad |\mu| = |\nu|\}.$$

□

It follows that $C^*(E)^\gamma = \Phi_\gamma(C^*(E))$.

Our next goal will be to prove that for any CK E -family $\{T, Q\}$, the $*$ -homomorphism $\pi_{T, Q}$ is injective on $C^*(E)^\gamma$. In order to do this, we will prove that $\pi_{T, Q}$ is injective on a space which contains $C^*(E)^\gamma$.

Definition 3.8. To this end, we define two new classes of subsets, where $k \geq 0$ of $C^*(E)$ to be

$$\begin{aligned} \mathcal{F}_k &= \overline{\text{span}}\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k, \quad s(\mu) = s(\nu)\}, \text{ and} \\ \mathcal{F}_k(v) &= \overline{\text{span}}\{s_\mu s_\nu^* \mid |\mu| = |\nu| = k, \quad s(\mu) = s(\nu) = v\}. \end{aligned}$$

It is not hard to verify that $\mathcal{F}_k = \oplus_{v \in E^0} \mathcal{F}_k(v)$. If we take paths μ, ν, α, β of equal length, then because ν and α cannot extend each other in a non-trivial way, $(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \delta_{\nu, \alpha} s_\mu s_\beta^*$, making the collection of $\{s_\mu s_\nu^*\} \cap \mathcal{F}_k(v)$ a family of matrix units for $\mathcal{F}_k(v)$.

Now, if the graph E does not contain sources and $\mu, \nu \in E^k \cap s^{-1}(v)$, then, using the CK relation, we have

$$s_\mu s_\nu^* = \sum_{r(e)=v} s_{\mu e} s_{\nu e}^*.$$

Hence $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$, giving the equality that

$$\begin{aligned} C^*(E)^\gamma &= \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \text{ and } |\mu| = |\nu|\} \\ &= \overline{\cup_{k \geq 0} \mathcal{F}_k} \\ &= \overline{\cup_{k \geq 0} (\oplus_{v \in E^0} \mathcal{F}_k(v))}. \end{aligned}$$

If the graph E does have sources, we need to take a different approach. Define the set

$$E^{\leq k} = \{\mu \in E^* \mid |\mu| = k \text{ or } |\mu| < k \text{ and } s(\mu) \text{ is a source}\}.$$

Note that, given paths $\alpha, \nu \in E^{\leq k}$ with $|\nu| < |\alpha|$, $s(\nu)$ is a source so α cannot extend ν . Thus, for $\mu, \alpha, \beta, \nu \in E^{\leq k}$ we have $(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \delta_{\nu, \alpha} s_\mu s_\beta^*$.

Definition 3.9. We define similar sets to those above, namely:

$$\begin{aligned}\mathcal{F}_{\leq k}(v) &= \overline{\text{span}}\{s_\mu s_\nu^* \mid \mu, \nu \in E^{\leq k} \quad s(\mu) = s(\nu) = v\} \\ \mathcal{F}_{\leq k} &= \overline{\text{span}}\{s_\mu s_\nu^* \mid \mu, \nu \in E^{\leq k}\}.\end{aligned}$$

Again, we have that $\mathcal{F}_{\leq k} = \oplus_{v \in E^0} \mathcal{F}_{\leq k}(v)$. If v is not a source, then for any $s_\mu s_\nu^* \in \mathcal{F}_{\leq k}(v)$,

$$s_\mu s_\nu^* = \sum_{\substack{r(e)=v \\ e \in \mathcal{F}_{\leq k+1}}} \underbrace{s_{\mu e} s_{\nu e}^*}_{\in \mathcal{F}_{\leq k+1}} \in \mathcal{F}_{\leq k+1}.$$

On the other hand, if v is a source, then any $s_\mu s_\nu^* \in \mathcal{F}_{\leq k}(v)$ is also contained in $\mathcal{F}_{\leq k+1}(v) \subseteq \mathcal{F}_{\leq k+1}$ so $\mathcal{F}_{\leq k} \subseteq \mathcal{F}_{\leq k+1}$. Thus, in the case that E has sources, we see that

$$C^*(E)^\gamma \subseteq \overline{\cup_{k \geq 0} \mathcal{F}_{\leq k}} = \overline{\cup_{k \geq 0} (\oplus_{v \in E^0} \mathcal{F}_{\leq k}(v))}.$$

We now have the definitions required to prove our lemma.

Lemma 3.10. *For a row-finite graph E and a CK E -family $\{T, Q\}$ in a C^* -algebra \mathfrak{B} such that $Q_v \neq 0$ for all $v \in E^0$, the $*$ -homomorphism $\pi_{T, Q}$ is injective on $C^*(E)^\gamma$.*

Proof. For every matrix unit $s_\mu s_\nu^*$ in $\mathcal{F}_{\leq k}(v)$ we have $\pi_{T, Q}(s_\mu s_\nu^*) = T_\mu T_\nu^*$. If this were equal to zero, then

$$\begin{aligned}T_\mu T_\nu^* = 0 &\Rightarrow T_\mu^* T_\mu T_\nu^* = 0 \\ &\Rightarrow T_\nu^* = 0 \\ &\Rightarrow T_\nu^* T_\nu = 0 \\ &\Rightarrow Q_{s(\nu)} = 0,\end{aligned}$$

which contradicts our assumption that the projections Q_v in $C^*(T, Q)$ are non-zero. Thus, the image of any matrix unit under $\pi_{T, Q}$ must be non-zero. Now, if $0 = \pi_{T, Q}(\sum a_{\mu\nu} s_\mu s_\nu^*) = \sum a_{\mu\nu} T_\mu T_\nu^*$, then, because $\{T_\mu T_\nu^* \mid \mu, \nu \in E^{\leq k}\}$ is a set of matrix units for $\mathcal{F}_{\leq k}(v)$, we can multiply on the left and right by $T_\alpha T_\alpha^*$ and $T_\lambda T_\lambda^*$ to get that $0 = a_{\alpha\lambda} T_\alpha T_\lambda^*$, forcing $a_{\alpha\lambda} = 0$ for all choices of $\alpha, \lambda \in E^{\leq k}$. This tells us that $\pi_{T, Q}$ is injective on each $\mathcal{F}_{\leq k}(v)$.

Thus, $\pi_{T, Q}$ is injective on $\mathcal{F}_{\leq k} = \oplus_v \mathcal{F}_{\leq k}(v)$. Because $\mathcal{F}_{\leq k}$ is a C^* -algebra, $\pi_{T, Q}$ is isometric on $\mathcal{F}_{\leq k}$ and so it's also isometric on $\cup_k (\oplus_v \mathcal{F}_{\leq k}(v))$. Since $\overline{\cup_k (\oplus_v \mathcal{F}_{\leq k}(v))}$ contains $C^*(E)^\gamma$, then $\pi_{T, Q}$ is also isometric on $C^*(E)^\gamma$. \square

This subsection has been leading up to this point so without further ado, we present the Gauge-Invariant Uniqueness Theorem:

Theorem 3.11. (Gauge-Invariant Uniqueness) *Suppose that $\{T, Q\}$ is a CK E -family in the C^* -algebra \mathfrak{B} with each $Q_v \neq 0$. If there is a continuous action $\beta : \mathbb{T} \rightarrow \text{Aut } \mathfrak{B}$ such that $\beta_z(T_e) = zT_e$ and $\beta_z(Q_v) = Q_v$ for all $e \in E^1$ and $v \in E^0$, then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Proof. Let γ be the gauge action on $C^*(E)$ defined in Theorem 3.2. Note first that $\pi_{T,Q} \circ \gamma_z = \beta_z \circ \pi_{T,Q}$ on $\{p_v, s_e\}$ and so they are equal for any $a \in C^*(E)$. Moreover, for the map Φ defined in Proposition 3.6, we have

$$\begin{aligned}
\|\pi_{T,Q}(\Phi(a))\| &= \left\| \pi_{T,Q} \left(\int_{\mathbb{T}} \gamma_z(a) dz \right) \right\| \\
&= \left\| \int_{\mathbb{T}} \pi_{T,Q}(\gamma_z(a)) dz \right\| && \text{(by (3) of lemma 3.4)} \\
&= \left\| \int_{\mathbb{T}} \beta_z(\pi_{T,Q}(a)) dz \right\| \\
&\leq \int_{\mathbb{T}} \|\beta_z(\pi_{T,Q}(a))\| dz && \text{(by (2) of lemma 3.4)} \\
&= \int_{\mathbb{T}} \|\pi_{T,Q}(a)\| dz && \text{(since automorphism are norm preserving)} \\
&= \|\pi_{T,Q}(a)\|.
\end{aligned}$$

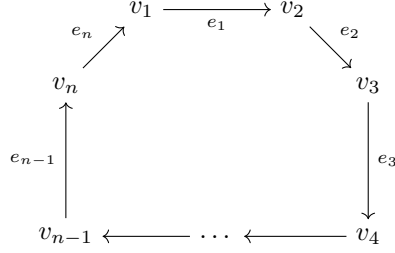
We will now show that $\pi_{T,Q}$ is injective:

$$\begin{aligned}
\pi_{T,Q}(a) = 0 &\Leftrightarrow \pi_{T,Q}(a^*a) = 0 \\
&\Leftrightarrow \pi_{T,Q}(\Phi(a^*a)) = 0 && \text{(by above argument)} \\
&\Leftrightarrow \Phi(a^*a) = 0 && \text{(because } \pi_{T,Q} \text{ is faithful on } C^*(E)^\gamma) \\
&\Leftrightarrow a^*a = 0 && \text{(by Proposition 3.6)} \\
&\Leftrightarrow a = 0.
\end{aligned}$$

Now, $\pi_{T,Q}(s_e) = T_e$ and $\pi_{T,Q}(p_v) = Q_v$ giving us that $\pi_{T,Q}(C^*(E))$ is generated by $\{T, Q\}$. Since the range of $\pi_{T,Q}$ must be a C^* -algebra, $\pi_{T,Q}(C^*(E)) = C^*(T, Q)$. Thus, $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$. \square

We may reword the theorem above as follows: Suppose for every CK E -family $\{T, Q\}$ in a C^* -algebra \mathfrak{B} with each $Q_v \neq 0$, there is an action $\beta : \mathbb{T} \rightarrow \text{Aut}(\mathfrak{B})$ such that $\beta_z \circ \pi = \pi \circ \gamma_z$ for all $z \in \mathbb{T}$. Then $\pi_{T,Q}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$. Let's look at how we may apply the Gauge-Invariant Uniqueness Theorem.

Example 3.12. Consider the graph C_n which is a cycle of n vertices, shown in Graph 7.



GRAPH 7

We will build a CK E -family $\{S, P\}$ in $\mathcal{C}(\mathbb{T}, M_n(\mathbb{C})) \cong M_n(\mathcal{C}(\mathbb{T}))$ by defining

$$P_{v_i}(z) = E_{i,i} \quad S_{e_i}(z) = E_{i+1,i} \text{ for } i < n \quad S_{e_n}(z) = zE_{1,n}.$$

We have the homomorphism $\pi_{S,P}$ from $C^*(C_n)$ onto $C^*(S, P)$. We may note that the range of $\pi_{S,P}$ contains all functions of the form $z \mapsto z^m E_{ij}$ where m is any integer. Indeed, we have that $E_{i1} = S_{e_{i-1}}(z) \dots S_{e_1}(z)$, so we may obtain the map $z \mapsto E_{i1}$. Moreover, the map $z \mapsto z^n E_{11}$ for $n \geq 1$ can be obtained by $S_{e_n}(z) \dots S_{e_2}(z) S_{e_1}(z)$ and so by combining these two maps and possibly taking adjoints, we have the map $z \mapsto E_{i1} z^m E_{11} E_{j1}^* = z^m E_{ij}$ for any integer m . By the Stone-Weierstrass Theorem, we get that $\text{ran } \pi_{S,P} = \mathcal{C}(\mathbb{T}, M_n(\mathbb{C}))$, and so $\pi_{S,P}$ is surjective. We are left to prove it is injective, and we will utilize the Gauge-Invariant Uniqueness Theorem for this step. To do so, we must construct an appropriate action β .

Fix some $w \in \mathbb{T}$ and define U_w to be the diagonal unitary matrix $\sum_{j=1}^n w^j E_{jj}$. Let

$$\beta_w(f)(z) = U_w f(w^n z) U_w^*.$$

Then we have

$$\beta_w(P_{v_i})(z) = \beta_w(E_{i,i})(z) = U_w E_{i,i} U_w^* = E_{i,i} U_w U_w^* = E_{i,i} = P_{v_i}(z).$$

For index $i < n$,

$$\begin{aligned} \beta_w(S_{e_i})(z) &= U_w E_{i+1,i} U_w^* \\ &= \sum_{j=1}^n \sum_{k=1}^n w^j E_{j,j} E_{i+1,i} w^{-k} E_{k,k} \\ &= w^{i+1} w^{-i} E_{i+1,i} \\ &= w S_{e_i}(z). \end{aligned}$$

Similarly, we have

$$\begin{aligned}
\beta_w(S_{e_n})(z) &= U_w E_{1,n}(w^n z) U_w^* \\
&= U_w(w^n z) E_{1,n} U_w^* \\
&= (s^n z)(w^1 w^{-n} E_{1,n}) \\
&= w z E_{1,n} \\
&= w S_{e_n}(z).
\end{aligned}$$

Thus, β is an appropriate action from \mathbb{T} to $\text{Aut } \mathcal{C}(\mathbb{T}, M_n(\mathbb{C}))$ where $\pi_{S,P} \circ \gamma_w = \beta_w \circ \pi_{S,P}$ for every $w \in \mathbb{T}$. The Gauge-Invariant Uniqueness Theorem tells us that $\pi_{S,P}$ is an isomorphism from $C^*(C_n)$ to $\mathcal{C}(\mathbb{T}, M_n(\mathbb{C}))$.

Since we know there exists the gauge action γ on $C^*(E)$, we may wonder why the homomorphism $\pi_{T,Q}$ from $C^*(E)$ to $C^*(T, Q)$ does not preserve the action to give us an appropriate action $\beta_z = \pi_{T,Q} \circ \gamma_z$ on $C^*(T, Q)$. We may instead ask if we can find some element $a \in \ker \pi_{T,Q} \subseteq C^*(E)$ such that $\gamma_z(a) \notin \ker \pi_{T,Q}$. The existence of such an element will prove that $\pi_{T,Q}$ is not injective since $\gamma_z(a) \notin \ker \pi_{T,Q}$ implies a is non-zero but $a \in \ker \pi_{T,Q}$ and so the Gauge-Invariant Uniqueness Theorem will tell us that an appropriate action β does not exist.

Consider Figure 6 which consisted of one vertex v and the single loop e at v . We have already shown that $C^*(E) = \mathcal{C}(\mathbb{T})$. We take the CK E -family $\{s, p\}$ to be $s_e(z) = e^{iz}$ so that $p_v(z) = s_e^*(z)s_e(z) = \overline{e^{iz}}e^{iz} = 1$ making $p_v = \mathbb{1}$. Then

$$C^*(S, P) = \overline{\left\{ \sum_{k=-N}^M a_k s_e^k \mid a_k \in \mathbb{Z} \right\}} = \mathcal{C}(\mathbb{T}),$$

telling us that $\{s, p\}$ is a universal CK E -family.

Now consider the CK E -family $\{T, Q\}$ given by $T_e = e^i \mathbb{1}$ and $Q_v = T_e^* T_e = \mathbb{1}$. $C^*(T, Q) = \overline{\text{span}\{e^i \mathbb{1}\}} = \mathbb{C}$, and by taking the homomorphism

$$\begin{aligned}
\pi : C^*(E) &\rightarrow C^*(T, Q) \\
s_e &\mapsto T_e = e^i \mathbb{1} \\
p_v &\mapsto Q_v = \mathbb{1},
\end{aligned}$$

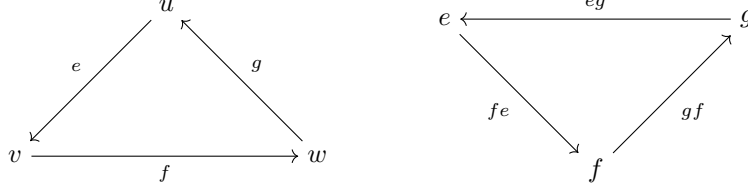
and the element $a = s_e - e^i p_v$, we see that $a \in \ker \pi$ but $\pi(\gamma_z(a)) = (z - 1)e^i \mathbb{1}$. Thus, $\gamma_z(a) \notin \ker \pi$ if z does not equal 1.

Consequently, such an action β does not always exist, making the assumption in Theorem 3.11 a non-trivial one.

3.2. Dual graphs. Before we proceed to a relatively straightforward application of the Gauge-Invariant Uniqueness Theorem, we define the notion of the dual graph. Note that this definition is different from the one commonly seen in graph theory where each face is replaced with a vertex and edges are added between vertices corresponding to adjacent faces.

Definition 3.13. Given a graph E , we can define the **dual graph**, \widehat{E} , by setting $\widehat{E}^0 = E^1$, $\widehat{E}^1 = E^2$ and modifying the maps r and s to be $r_{\widehat{E}}(ef) = e$ and $s_{\widehat{E}}(ef) = f$.

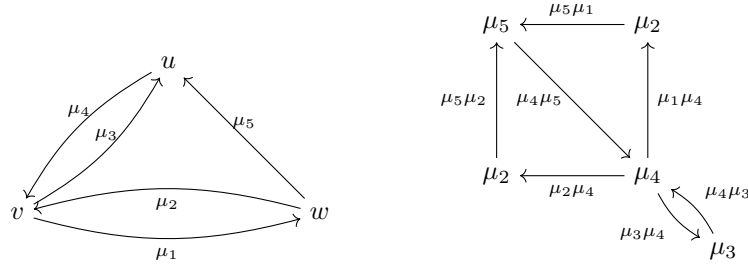
Example 3.14. The easiest way to understand the definition is to look at a few examples:



GRAPH 8. the original graph E on the left with its dual on the right.

Here, we have the original graph on the left and the dual graph on the right. In this case, the graph on the right is just a relabeled version of the original graph. That is, the original graph and the dual graph are isomorphic to one another.

Example 3.15. This is not always the case, as we can see below:



GRAPH 9. the original graph E on the left with its dual on the right.

Again, we have the original graph on the left and the dual graph on the right.

We have the following result regarding dual graphs.

Theorem 3.16. For a row-finite graph E with no sources, \widehat{E} is also row-finite and $C^*(\widehat{E}) \cong C^*(E)$.

Proof. For any $e \in \widehat{E}^0 = E^1$, we have that

$$\#r_{\widehat{E}}^{-1}(e) = \#\{ef \in E^2 = \widehat{E}^1\} = \#\{f \mid s_E(e) = r_E(f)\} = \#r_E^{-1}(s_E(e)).$$

Since E is row-finite, then $\#r_{\widehat{E}}^{-1}(e) = \#r_E^{-1}(s_E(e)) < \infty$ so that \widehat{E} is also row-finite. Now let $\{s, p\}$ be the universal CK E -family which generates $C^*(E)$. We may define the following CK \widehat{E} -family:

$$Q_e = s_e s_e^* \quad T_{fe} = s_f s_e s_e^*, \quad e, f \in E^1.$$

Since $s_e^* s_f \neq 0$ implies $e = f$, this tells us that $Q_e Q_f = s_e s_e^* s_f s_f^* = 0$ for $e \neq f$. Moreover, $Q_e^* = s_e s_e^* = Q_e$ and $Q_e^2 = s_e s_e^* s_e s_e^* = s_e p_{s(e)} s_e^* = s_e s_e^* = Q_e$ so that the collection $\{Q_e\}_e$ forms a set of mutually orthogonal projections. We now check the CK-conditions. First, for $fe \in \widehat{E}^1$,

$$\begin{aligned} T_{fe}^* T_{fe} &= (s_f s_e s_e^*)^* (s_f s_e s_e^*) \\ &= s_e s_e^* (s_f^* s_f) s_e s_e^* \\ &= s_e s_e^* Q_{s(f)} s_e s_e^* \\ &= s_e s_e^* \\ &= Q_{s(fe)}, \end{aligned}$$

proving the T_{fe} are partial isometries. Finally,

$$\begin{aligned} Q_f &= s_f s_f^* \\ &= s_f \left(\sum_{r(e)=s(f)} s_e s_e^* \right) s_f^* \\ &= \sum_{r(e)=s(f)} s_f (s_e s_e^*) (s_e s_e^*) s_f^* \\ &= \sum_{r(fe)=f} T_{fe} T_{fe}^*. \end{aligned}$$

Thus, the collection $\{Q, T\}$ is a CK \widehat{E} -family. Now suppose that $\{t, q\}$ generates the universal C^* -algebra $C^*(\widehat{E})$. By the universal property of $C^*(\widehat{E})$, there is a homomorphism $\pi_{T, Q} : C^*(\widehat{E}) \rightarrow C^*(T, Q)$ which sends t_{fe} to T_{fe} and q_v to Q_v .

Since the operators Q_e, T_{fe} were defined from $\{s, p\}$, we have $C^*(T, Q) \subseteq C^*(E)$. Note that we can also recover the CK E -family $\{s, p\}$ via $p_v = \sum_{r(e)=v} Q_e$ (this applies for each $v \in E^0$ since E contains no sources) and $s_f = \sum_{s(f)=r(e)} T_{fe}$, which implies that we have $C^*(T, Q) = C^*(E)$. Since $s_e \neq 0$ for all $e \in E^1$, $Q_e = s_e s_e^* \neq 0$. We know that there exists a gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(E) = \text{Aut } C^*(T, Q)$ which satisfies $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = z s_e$. Thus, γ_z will then satisfy $\gamma_z(Q_e) = \gamma_z(s_e) \gamma_z(s_e^*) = z s_e z^* s_e^* = s_e s_e^* = Q_e$ and $\gamma_z(T_{fe}) = \gamma_z(s_f s_e s_e^*) = z s_f z s_e z^* s_e^* = z s_f s_e s_e^* = z T_{fe}$. We can now apply the Gauge-Invariant Uniqueness Theorem to find that $\pi_{T, Q}$ is an isomorphism of $C^*(\widehat{E})$ onto $C^*(T, Q) = C^*(E)$. \square

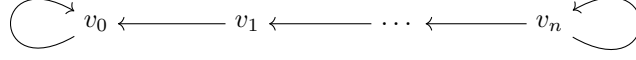
In general, if E has no sources, then \widehat{E} also has no sources. Indeed, if \widehat{E} has a source e then there is no edge ef in \widehat{E} . That is, there is no edge f in E with $r(f) = s(e)$ and thus, $s(e)$ is a source in E . Hence, we may repeatedly apply Theorem 3.16.

Example 3.17. Consider the graph E , its dual \widehat{E} and its second dual $\widehat{\widehat{E}}$ given in Graph 10.

$$e \curvearrowright v \xleftarrow{f} w \curvearrowright g \quad ee \curvearrowright e \xleftarrow{ef} f \xleftarrow{fg} g \curvearrowright gg \quad eeee \curvearrowright ee \xleftarrow{eeef} ef \xleftarrow{effg} fg \xleftarrow{fggg} gg \curvearrowright gggg$$

GRAPH 10. the original graph E on the left, its dual in the center and its double dual on the right.

Since we can continue to take duals, we find that $C^*(E) \cong C^*(\widehat{E}) \cong C^*(\widehat{\widehat{E}}) \cong C^*(F)$ where F is the n^{th} dual of E . Note that the n^{th} dual of E would look like Graph 11.

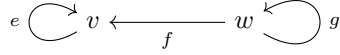


GRAPH 11

Thus, the C^* -algebra of any graph of the form above with $n \geq 1$ will be isomorphic to $C^*(E)$. We will now find what $C^*(E)$ is. For $0 \leq q < 1$, let $SU_q(2)$ be the universal C^* -algebra generated by elements a and b (see [13] for existence) which satisfy

$$a^*a + b^*b = 1 \quad aa^* + q^2b^*b = 1 \quad ab = qba \quad ab^* = qb^*a \quad b^*b = bb^*.$$

Label the graph E as in Graph 12.



GRAPH 12

We have the CK relations

$$p_v = s_f^*s_f = s_e^*s_e = s_e s_e^* \quad p_w = s_g^*s_g = s_g s_g^* + s_f s_f^*.$$

It is already known that for any $0 \leq q < 1$ the spaces $SU_q(2)$ are isomorphic (see [13]). We further claim that any $SU_q(2)$ is isomorphic to $C^*(E)$. For simplicity, we will show that $C^*(E)$ and $SU_0(2)$ are isomorphic.

Indeed, define the mappings

$$\begin{aligned} \varphi : SU_0(2) &\rightarrow C^*(E) \\ a &\mapsto s_f^* + s_g^* \\ b &\mapsto s_e \end{aligned}$$

$$\begin{aligned} \phi : C^*(E) &\rightarrow SU_0(2) \\ p_v &\mapsto bb^* \\ p_w &\mapsto a^*a \\ s_e &\mapsto b \\ s_g &\mapsto a^*(1 - bb^*) \\ s_f &\mapsto a^*bb^*. \end{aligned}$$

We can check that $\varphi(a)$ and $\varphi(b)$ satisfy the relations of $SU_0(2)$ and that ϕ preserves the CK-relations. Finally, it is easy to check that ϕ is the inverse of φ and so we have defined an isomorphism from $SU_0(2)$ to $C^*(E)$, making the two spaces are isomorphic.

Note that if there is a cycle in E then there is a cycle in \widehat{E} . Indeed, suppose $\mu_1\mu_2\ldots\mu_n$ is a cycle in E . Then we have $\{\mu_i\mu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\mu_n\mu_1\} \in E^2 = \widehat{E}^1$ so that $\mu = (\mu_1\mu_2)(\mu_2\mu_3)\ldots(\mu_{n-1}\mu_n)(\mu_n\mu_1) \in \widehat{E}^*$ and $s(\mu) = \mu_1 = r(\mu)$ making μ a cycle in \widehat{E} .

Thus, by virtue of the fact that every finite directed graph with no sources has a cycle, we cannot reduce such a graph to the case where the assumptions in Proposition 2.14 are satisfied.

3.3. The CK-Uniqueness Theorem. Our next goal will be to prove the CK-Uniqueness Theorem. This theorem gives the powerful result that under certain conditions the C^* -algebra generated by any two CK-families is isomorphic. We first present a lemma which will be utilized in the proof of the CK-Uniqueness Theorem. Because the proof does not introduce any new techniques and is somewhat long, we skip the proof and refer the reader to [10]. Recall that the **multiplier algebra** $M(\mathfrak{A})$ of a C^* -algebra \mathfrak{A} is the unique C^* -algebra with the property that $M(\mathfrak{A})$ is the maximal unital extension of \mathfrak{A} for which \mathfrak{A} is an essential ideal.

Lemma 3.18. *Given a row-finite graph E and a set of vertices $V \subseteq E^0$ (which may be either finite or infinite), there exists a projection $p_V = \sum_{v \in V} p_v$ in $M(C^*(E))$ such that*

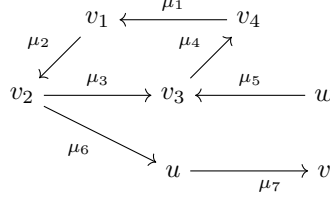
$$p_V s_\mu s_\nu^* = \begin{cases} s_\mu s_\nu^* & \text{if } r(\mu) \in V \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.19. We say that an edge e is an **entry** to the cycle $\mu = \mu_1 \ldots \mu_n$ if there exists an index i such that $r(e) = r(\mu_i)$ and $e \neq \mu_i$. In particular, if we have a closed path with no entries then it is guaranteed to be a multiple of some cycle ν with no entries. We call a path $\mu \in E^*$ **non-returning** if $\mu_k \neq \mu_{|\mu|}$ for $k < |\mu|$.

Lemma 3.20. *Suppose the row-finite graph E has no sources and every cycle in E has an entry. Then for every vertex v and any positive integer n , there exists a non-returning path $\lambda \in E^*$ of length at least n with $r(\lambda) = v$.*

Proof. If there is path $\lambda \in E^n$ with $r(\lambda) = v$ and no repeated vertices, then this concludes the proof. If not, then every path of length n which ends at v contains a return path and so we can choose the shortest return path α such that $r(\alpha) = v$ and there is a cycle β based at $s(\alpha)$. The assumption implies that β has an entry e , so for sufficiently many repetitions of β , the path $\lambda = \alpha\beta\beta\ldots\beta\beta'e$ has the required properties, where β' is the segment of β from $r(e)$ to $s(\beta)$. \square

Example 3.21. Given the following graph, suppose we want a non-returning path λ of length n with $r(\lambda) = v$ in Graph 13.



GRAPH 13

For sufficiently large n we cannot find a path with no repeated vertices, but any path of sufficient length will contain the cycle $\beta = \mu_2\mu_1\mu_4\mu_3$. The entry e is given by μ_5 and we have that $\alpha = \mu_7\mu_6$. Thus, we can set the path to be

$$\lambda = \alpha\beta\beta\ldots\beta\beta'\mu_5 = \underbrace{(\mu_7\mu_6)}_{\alpha} \underbrace{(\mu_2\mu_1\mu_4\mu_3)}_{\beta} \ldots \underbrace{(\mu_2\mu_1\mu_4\mu_3)}_{\beta} \underbrace{(\mu_2\mu_1\mu_4)}_{\beta'} \mu_5.$$

Note that if we repeat β n times then the path will certainly be of length at least n .

We now present the CK-Uniqueness Theorem:

Theorem 3.22. (CK-Uniqueness) *Consider the row-finite graph E where every cycle has an entry. Let $\{T, Q\}$ be a CK E -family in a C^* -algebra \mathfrak{B} such that $Q_v \neq 0$ for every vertex v . Then the homomorphism $\pi_{T, Q} : C^*(E) \rightarrow \mathfrak{B}$ is an isomorphism of $C^*(E)$ onto $C^*(T, Q)$.*

Proof. The case where E has a source follows from the theory of an extended graph E_+ which is obtained from E by adding a sequence of edges into every source and a sequence of edges from every sink. The rigorous proof may be found in [10]. For the purposes of this report, we will only consider the case where E has no sources. Fix some finite set F of pairs $(\mu, \nu) \in E^* \times E^*$ where $s(\mu) = s(\nu)$ and element $a = \sum_{(\mu, \nu) \in F} c_{\mu\nu} s_\mu s_\nu^*$. Recall that we have $\Phi(a) = \int_{\mathbb{T}} \gamma_z(a) dz$ as defined in Theorem 3.6. Suppose we could show that there exists a projection $Q \in \mathfrak{B}$ satisfying

$$\|Q\pi_{T, Q}(\Phi(a))Q\| = \|\pi_{T, Q}(\Phi(a))\|, \quad \text{and} \quad QT_\mu T_\nu^* Q = 0 \text{ when } (\mu, \nu) \in F \text{ and } |\mu| \neq |\nu|.$$

If such a projection Q could be found, then

$$\begin{aligned}
\|\pi_{T,Q}(\Phi(a))\| &= \|Q\pi_{T,Q}(\Phi(a))Q\| \\
&= \left\| Q \left(\sum_{(\mu,\nu) \in F, |\mu|=|\nu|} c_{\mu\nu} T_\mu T_\nu^* \right) Q \right\| \\
&= \left\| Q \left(\sum_{(\mu,\nu) \in F} c_{\mu\nu} T_\mu T_\nu^* \right) Q \right\| \\
&\leq \left\| \sum_{(\mu,\nu) \in F} c_{\mu\nu} T_\mu T_\nu^* \right\| \\
&= \|\pi_{T,Q}(a)\|.
\end{aligned}$$

Thus, if $\pi_{T,Q}(a) = 0$, then because $\pi_{T,Q}$ is a *-homomorphism, $\pi_{T,Q}(a^*a) = 0$. It follows that $\pi_{T,Q}(\Phi(a^*a)) = 0$, so $\Phi(a^*a) = 0$ implies a must be zero. Hence, $\pi_{T,Q}$ is an isomorphism.

In order to prove such a projection exists, let's first set $k = \max\{|\mu|, |\nu| \mid (\mu, \nu) \in F\}$. Next, suppose that $c_{\mu\nu} \neq 0$ for $(\mu, \nu) \in F$ and let $r^{-1}(s(\mu)) = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then

$$s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* = \sum_{i=1}^n s_{\mu\alpha_i} s_{\nu\alpha_i}^*.$$

By replacing each (μ, ν) in F with $\{(\mu\alpha_i, \nu\alpha_i)\}_{i=1}^n$ we can modify the length of each pair to force $\max\{|\mu|, |\nu|\}$ to equal k for any $(\mu, \nu) \in F$ with $c_{\mu\nu} \neq 0$. In particular, if $|\mu| = |\nu|$ and $c_{\mu\nu} \neq 0$ then $|\mu| = |\nu| = k$.

We've already shown that $\Phi(a) \in \mathcal{F}_k = \oplus_v \mathcal{F}_k(v)$ and so for some vertex w , we have

$$\|\Phi(a)\| = \|b_w\| \quad \text{where} \quad b_w := \sum_{\substack{(\mu,\nu) \in F \\ |\mu|=|\nu| \\ s(\mu)=s(\nu)=w}} c_{\mu\nu} s_\mu s_\nu^*.$$

Let $G = \{(\mu, \nu) \mid (\mu, \nu) \in F, |\mu| = |\nu| = k, s(\mu) = s(\nu)\}$. For the vertex w and $n > \max\{|\mu|, |\nu| \mid (\mu, \nu) \in F\}$ choose some non-returning path λ as in Lemma 3.20, and let

$$Q := \sum_{\tau \in G} T_{\tau\lambda} T_{\tau\lambda}^*.$$

We now check that this Q satisfies our criteria. Indeed, if $(\mu, \nu) \in F$ satisfies $|\mu| = |\nu|$ then $\mu, \nu \in G$. Since for $\tau \in G$, we have that $T_{\tau\lambda}^* T_\mu$ is non-zero if and only if $\tau = \mu$, which implies $T_\nu^* T_{\tau\lambda}$ is non-zero if and only if $\tau = \nu$.

Thus,

$$\begin{aligned}
QT_\mu T_\nu^* Q &= \sum_{\alpha \in G} \sum_{\tau \in G} T_{\tau\lambda} T_{\tau\lambda}^* T_\mu T_\nu^* T_{\alpha\lambda} T_{\alpha\lambda}^* \\
&= \begin{cases} T_{\mu\lambda} T_{\mu\lambda}^* T_\mu T_\nu^* T_{\nu\lambda} T_{\nu\lambda}^* & \text{if } \mu, \nu \in G \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} T_{\mu\lambda} T_{\nu\lambda}^* & \text{if } \mu, \nu \in G \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For any path τ in G , since $T_{\tau\lambda}^* T_{\tau\lambda} = Q_{s(\tau\lambda)} \neq 0$ then $T_{\tau\lambda} T_{\tau\lambda}^* \neq 0$ as well, making the set $\{QT_\mu T_\nu^* Q \mid \mu, \nu \in G\}$ a collection of non-zero matrix units. Because the mapping $b \mapsto Q\pi_{T,Q}(b)Q$ is a faithful representation (hence isometric mapping) of $\text{span}\{s_\mu s_\nu^* \mid \mu, \nu \in G\}$,

$$\|\pi_{T,Q}(\Phi(a))\| = \|\Phi(a)\| = \|b_w\| = \|Q\pi_{T,Q}(b_w)Q\| = \|Q\pi_{T,Q}(\Phi(a))Q\|.$$

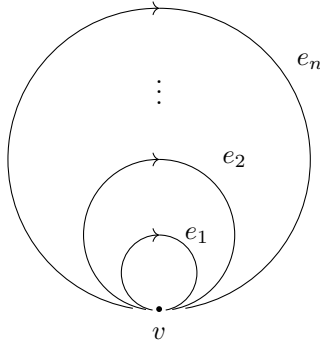
Now, if $(\mu, \nu) \in F$ with $|\mu| \neq |\nu|$, then the longer one must have length k . We will assume without loss of generality that μ is longer than ν , so $|\mu| = k$. Then

$$QT_\mu T_\nu^* Q = \sum_{\tau, \alpha \in G} T_{\alpha\lambda} T_{\alpha\lambda}^* T_\mu T_\nu^* T_{\tau\lambda} T_{\tau\lambda}^* = \sum_{\tau \in G} T_{\mu\lambda} (T_{\nu\lambda}^* T_{\tau\lambda}) T_{\tau\lambda}^*.$$

If $T_{\nu\lambda}^* T_{\tau\lambda} \neq 0$ then $\nu\lambda$ extends $\tau\lambda$ or vice versa. However, because $0 < |\nu| - |\tau| < |\lambda|$ and λ is non-returning, this cannot happen. So if $|\mu| \neq |\nu|$ then $QT_\mu T_\nu^* Q = 0$. Thus, we have the desired Q , proving the theorem in the case where E has no sources. \square

Note that the CK-Uniqueness Theorem implies that, for a graph E with every cycle having an entry and for any two CK E -families $\{S, P\}$ and $\{T, Q\}$ with $P_v \neq 0$ and $Q_v \neq 0$ for all vertices v , there is an isomorphism ϕ of $C^*(S, P)$ onto $C^*(T, Q)$ such that $\phi(S_e) = T_e$ and $\phi(P_v) = Q_v$.

Example 3.23. We now examine the graph with a single vertex v and $n \geq 2$ loops all at v as shown in Graph 14.

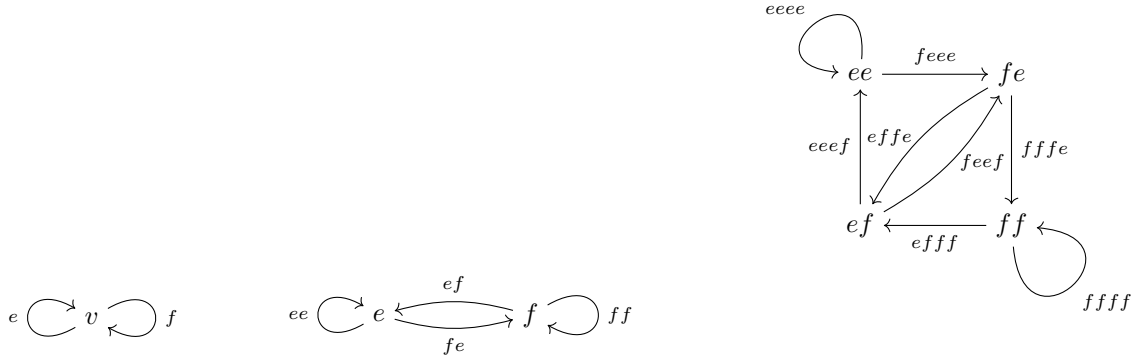


GRAPH 14

Since cycles cannot repeat vertices, the only cycles are each edge. Since a particular edge is an entry for the others, each cycle has an entry and so we may apply the CK-Uniqueness Theorem.

Note that for any edge e , we have $s_e = p_v s_e = s_e p_v$ so that p_v is an identity of $C^*(E)$. Then any two families $\{s_i\}$ of isometries such that $\sum_{i=1}^n s_i s_i^* = p_v = \mathbf{1}$ and $s_i^* s_i = \mathbf{1}$ generate isomorphic C^* -algebras, and this is (by definition) the Cuntz algebra \mathcal{O}_n . This was originally done by Cuntz in [3].

Now because the Cuntz algebra has no sources, we can apply Theorem 3.16 to the corresponding graph and find that we have the duals shown in Graph 15.



GRAPH 15. the original graph is shown on the left, its first dual in the center and the second dual on the right

While the double dual of the graph isn't obvious from the first graph, we can always find the first dual of the graph with a single vertex with n loops. It will consist of n vertices corresponding to e_1, e_2, \dots, e_n , there will be a loop at each vertex and there will be two edges between any pair of vertices $e_i e_j$ and $e_j e_i$. Theorem 3.16 tells us that the C^* -algebra of this graph will also be \mathcal{O}_n .

We now present a corollary of a previous proposition which gives us a way to know if a cycle has an entry.

Corollary 3.24. *Suppose we have a row-finite graph E with cycle μ . Then μ is entryless if and only if $S_\mu S_\mu^* = P_{r(\mu)}$.*

Proof. We first note that the cycle μ has an entry if and only if there exists a distinct path λ with $r(\lambda) = r(\mu)$ such that one of

- (1) $|\lambda| < |\mu|$ and $s(\lambda)$ is a source
- (2) $|\lambda| = |\mu|$.

Then from Proposition 2.10, and because $S_\lambda S_\lambda^* \neq 0$, we have

$$\begin{aligned}
 P_{r(\mu)} &= \sum_{\substack{r(\nu)=r(\mu), \\ |\nu|=|\mu|}} S_\nu S_\nu^* + \sum_{\substack{r(\nu)=r(\mu), \\ |\nu|<|\mu| \text{ and} \\ s(\nu) \text{ is a source}}} S_\nu S_\nu^* \\
 &\geq S_\mu S_\mu^* + S_\lambda S_\lambda^* \quad (\text{first sum includes } S_\mu S_\mu^* \text{ and second includes } S_\lambda S_\lambda^*) \\
 &\geq S_\mu S_\mu^*.
 \end{aligned}$$

Thus, the statement holds. □

4. IDEAL STRUCTURE OF $C^*(E)$

We will now examine the ideal structure of $C^*(E)$ and completely classify when $C^*(E)$ is simple.

Definition 4.1. For vertices v and w , we write $w \leq v$ if there exists a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. Note that this is transitive and reflexive but it is not a partial order because $v \leq w$ and $w \leq v$ do not imply that the two vertices are equal. We define the two sets E^∞ and $E^{\leq\infty}$ as

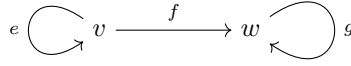
$$E^\infty = \{ \text{infinite paths } \lambda = \lambda_1 \lambda_2 \dots \} \text{ and}$$

$$E^{\leq\infty} = E^\infty \cup \{ \text{finite paths beginning at a source} \}.$$

Given a path $\mu \in E^* \cup E^\infty$, we define $[\mu]$ to be the set of vertices visited by μ .

Definition 4.2. We call a graph E **cofinal** if for every $\mu \in E^{\leq\infty}$ and $v \in E^0$ there exists a vertex $w \in [\mu]$ such that $v \leq w$.

Example 4.3. Consider Graph 12 studied in Section 3.2, shown again below.



Because the graph E does not have any sources, then

$$E^{\leq\infty} = E^\infty = \{eee\dots\} \cup \{g^k f e e e\dots \mid k \in \mathbb{N}_0\} \cup \{ggg\dots\}.$$

We cannot reach v from any point on the path $ggg\dots$, and so the graph above is not cofinal. However, if we were to remove the edge g then $E^{\leq\infty}$ would contain only the two infinite paths $eee\dots$ and $f e e e\dots$ and so the graph with the edge g removed would be cofinal.

Theorem 4.4. Suppose E is a row-finite graph in which every cycle has an entry. If E is cofinal, then $C^*(E)$ is simple.

Proof. First, we claim that every ideal in $C^*(E)$ is the kernel of a CK-family representation. Indeed, if I is the kernel of $C^*(E)$ then we can consider the quotient map $q : C^*(E) \rightarrow C^*(E)/I =: \mathfrak{A}$. Because \mathfrak{A} is also a C^* -algebra, we can find a $*$ -isomorphism $\phi : C^*(E)/I \rightarrow \mathcal{B}(\mathcal{H})$. Now $\pi = \phi \circ q : C^*(E) \rightarrow \mathcal{B}(\mathcal{H})$ is a representation with $\ker \pi = \ker q = I$. It sends s_e to $T_e := \pi(s_e) \in \mathcal{B}(\mathcal{H})$ and p_v to some $Q_v := \pi(p_v) \in \mathcal{B}(\mathcal{H})$. These satisfy,

$$T_e^* T_e = \pi(s_e^* s_e) = \pi(p_{s(e)}) = Q_{s(e)} \quad \sum_{r(e)=v} T_e T_e^* = \pi \left(\sum_{r(e)=v} s_e s_e^* \right) = \pi(p_v) = Q_v,$$

so $\{T, Q\}$ is a CK E -family. Now, the representation π is equal to $\pi_{T,Q}$, hence $I = \ker \pi_{T,Q}$. Thus, it suffices to prove that every non-zero representation $\pi_{T,Q}$ of $C^*(E)$ is faithful.

Suppose $\{s, p\}$ is a CK E -family such that $\pi_{s,p}$ is non-zero. If every projection p_v were zero, then $s_e^* s_e = p_{s(e)} = 0$ would force $s_e = 0$ for each $e \in E^1$ making $\pi_{s,p}$ identically zero. Hence, we can fix some vertex v such that $p_v \neq 0$. To see the kernel is trivial, we will show that $p_w \neq 0$ for all vertices w . This will allow us to apply the CK-Uniqueness Theorem to see that the representation $\pi_{T,Q}$ is faithful, and hence has no kernel.

Fix an arbitrary vertex $v \in E^0$. If v is not a source, then because $0 \neq p_v = \sum_{r(e)=v} s_e s_e^*$, there exists an edge e such that $r(e) = v$ and $s_e s_e^* \neq 0$. Thus, $p_{s(e)} = s_e^* s_e \neq 0$ and if $s(e)$ is not a source, we may repeat this argument by finding a second edge. Either this process will terminate at a source or it will continue indefinitely to give us an infinite path. In either case, we have constructed a path $\mu \in E^{\leq \infty}$ with $r(\mu) = v$ and $p_x \neq 0$ for every vertex x on μ .

Since E is cofinal, for every vertex w , there exists a path $\alpha \in E^*$ with $r(\alpha) = w$ and $s(\alpha)$ a vertex on μ . Then $s_\alpha^* s_\alpha = p_{s(\alpha)} \neq 0$, so $s_\alpha s_\alpha^* \neq 0$. Because $p_w s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \neq 0$, we have $p_w \neq 0$. Thus, $p_w \neq 0$ for every $w \in E^0$ and so by the CK-Uniqueness Theorem, the CK E -family $\{s, p\}$ generates $C^*(E)$ and $\pi_{s,p}$ is an isomorphism. In particular, $\pi_{s,p}$ is faithful. \square

Definition 4.5. A graph E is called **transitive** all vertices v and w satisfy $v \leq w$ and $w \leq v$.

Proposition 4.6. Suppose E is a row-finite, transitive graph which is not a cycle itself. Then $C^*(E)$ is simple.

Proof. Suppose we have such a graph E . Clearly, E is cofinal.

We claim that every cycle in E has an entry. For an arbitrary cycle μ and any edge e not in the cycle, if e is an entry, then the result holds. If e is not an entry into μ then because $r(e) \geq v$ for every vertex v in the cycle, there exists a path $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ from $r(e)$ to v . Moreover, by possibly truncating α we can assume that α is a path from $r(e)$ to the cycle μ where no edge in α is an edge in μ . The edge α_1 is an entry to the cycle μ , proving that every cycle in such a graph E has an entry. Thus, the C^* -algebra for any transitive graph which is not a cycle itself is simple by Theorem 4.4. \square

We now consider the case where $C^*(E)$ is not simple.

Definition 4.7. Suppose I is an ideal of $C^*(E)$. Let

$$H_I = \{v \in E^0 \mid p_v \in I\}.$$

Under certain conditions, we claim that we can recover the ideal I from the collection of vertices H_I . Let's investigate what these conditions are.

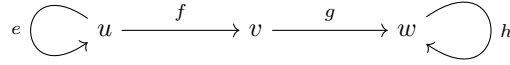
Consider the quotient map $q : C^*(E) \rightarrow C^*(E)/I$. If $v \notin H_I$ then $q(p_v) \neq 0$. For any edge e , if $s(e) \notin H_I$ then $q(s_e)^* q(s_e) = q(s_e^* s_e) = q(p_{s(e)}) \neq 0$. Now because $p_{r(e)} \geq s_e s_e^*$, then $q(p_{r(e)}) \geq q(s_e s_e^*) \neq 0$. Thus, $r(e) \notin H_I$.

Definition 4.8. For such a collection H_I , define

$$E \setminus H_I = (E^0 \setminus H_I, s^{-1}(E^0 \setminus H_I), r, s).$$

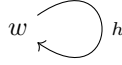
We have already shown that if the source of an edge is not in H_I then neither is the range, and so $E \setminus H_I$ is a well-defined graph. Moreover, $\{q(s_e), q(p_v) \mid s(e) \notin H_I, v \notin H_I\}$ is a CK-family for $E \setminus H_I$ with every vertex projection non-zero. If every cycle in the graph $E \setminus H_I$ has an entry, then we may apply the CK-Uniqueness Theorem to conclude that the map from $C^*(E \setminus H_I)$ onto $C^*(q(s_e), q(p_v)) = C^*(E) \setminus I$ is an isomorphism.

Example 4.9. Take the graph E to be as in Graph 16.



GRAPH 16

Let's take a look at the collection $H_I = \{u, v\}$, where we will assume for now that it is of the form presented above for some ideal I in $C^*(E)$. We see that $E \setminus H_I$ looks like Graph 17.



GRAPH 17

In this case, there is only one cycle and it does not contain an entry, so we may not apply the CK-Uniqueness Theorem as described above.

Our next goal is to identify potential collections $H \subseteq E^0$ which arise as sets H_I for some ideal I in $C^*(E)$, and find a condition on the graph E to guarantee that every cycle in $E \setminus H$ has an entry.

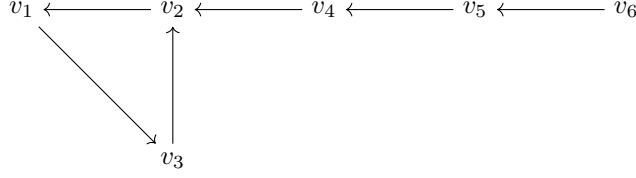
Definition 4.10. We call a subset $H \subseteq E^0$ **hereditary** if $w \in H$ and $w \leq v$ imply $v \in H$. We call H **saturated** if for any vertex $v \in E^0$ where v is not a source and $\{s(e) \mid r(e) = v\} \subseteq H$ imply $v \in H$.

Example 4.11. We revisit Graph 16. The set $\{u\}$ is hereditary since there are no vertices $v \neq u$ which satisfy $v \geq u$. The set $\{u, v\}$ is hereditary since we do not have $w \geq v$ nor $w \geq u$, and thus there is no assumption that w must also be included in the hereditary set containing u and v . Note that $\{v\}$ is not hereditary since $v \leq u$ but $u \notin \{v\}$.

We now consider which collections of vertices are saturated. Since $\{s(e) \mid r(e) = w\} \not\subseteq \{u, v\}$ then $\{u, v\}$ is saturated. The set $\{u\}$ is not saturated since $v \notin \{u\}$ is not a source but satisfies $\{s(e) \mid r(e) = v\} \subseteq \{u\}$.

It may be natural to wonder whether every saturated set is necessarily hereditary. This is not the case, as the next example shows.

Example 4.12. Consider Graph 18.



GRAPH 18

We can verify that the hereditary sets of E are $\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{v_4, v_5, v_6\}, \{v_5, v_6\}, \{v_6\}, \emptyset$ and the saturated sets are $\{v_1, v_2, v_3\}, \{v_6, v_5, v_4\}, \{v_1, v_2, v_3, v_4, v_5, v_6\}, \emptyset$.

Further note that for a source w , $\{w\}$ is guaranteed to be hereditary but not necessarily saturated. It is not hard to see that both E^0 and \emptyset are always saturated and hereditary, we call these the **trivial sets**.

Definition 4.13. For any saturated, hereditary collection H , we define $E \setminus H$ to be the graph $(E^0 \setminus H, s^{-1}(E^0 \setminus H), r, s)$.

Lemma 4.14. *Given a row-finite graph E , suppose I is a non-zero ideal in the C^* -algebra of E , $C^*(E)$. Then the set H_I is both saturated and hereditary.*

Proof. We first see that H_I is hereditary. Take some vertex $w \in H_I$ and suppose we have $w \leq v$ for the vertex v . Choose the appropriate path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. Since $w \in H_I$, we have $p_w \in I$ and $s_\mu = p_{r(\mu)} s_\mu = p_w s_\mu \in I$. So $p_v = s_\mu^* s_\mu \in I$ as well. Thus, $v \in H_I$ and so H_I is hereditary.

We will now see that H_I is saturated. Suppose v is not a source and that $\{s(e) \mid r(e) = v\} \subseteq H_I$. For every edge e with $r(e) = v$ we know $s_e = s_e p_{s(e)} \in I$. Since v is not a source, $p_v = \sum_{r(e)=v} s_e s_e^* \in I$ and so $v \in H_I$. Thus, H_I is saturated. \square

Definition 4.15. For a path μ , let $[\mu]$ be the set of all vertices visited by μ .

Definition 4.16. We will now define a graph E to satisfy **condition (K)** if for every vertex $v \in E^0$ either

- (1) there is no cycle λ with $r(\lambda) = v = s(\lambda)$, or
- (2) there are two distinct paths $\mu, \nu \in E^*$ such that $s(\mu) = r(\mu) = s(\nu) = r(\nu) = v$ with $r(\mu_i) \neq v$ for $i < |\mu|$ and $r(\nu_j) \neq v$ for $j < |\nu|$.

Example 4.17. Consider Graph 19.

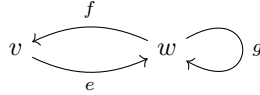


GRAPH 19

The graph on the left does not satisfy condition (K) at w since we are looking for return paths and thus, cannot find a distinct closed path at w which does not repeat e . However, it is satisfied at vertices u and v . By adding a second loop at vertex w to obtain the graph on the right, we see that condition (K) is satisfied.

It is important to note that we do not require the path $\mu, \nu \in E^*$ to be cycles because there is no stipulation regarding repeated vertices.

Example 4.18. To this end, Graph 20 satisfies condition (K). Indeed, at vertex w we have paths $\mu = g$ and $\nu = ef$. At the vertex v we have the paths $\mu = fe$ and $\nu = fge$.



GRAPH 20

Proposition 4.19. *The row-finite graph E satisfies condition (K) if and only if for every saturated, hereditary subset $H \subset E^0$, every cycle in $E \setminus H$ has an entry.*

Proof. To see the forward direction, let H be a saturated and hereditary subset of E^0 and suppose μ is a cycle in $E \setminus H$. Set $v = s(\mu) \notin H$. Condition (K) says that there exists a distinct path ν with $s(\nu) = v$. Choose i such that $\mu_j = \nu_j$ for $j < i$ and $\mu_i \neq \nu_i$.

Since $r(\nu_i) = s(\nu_{i-1}) = s(\mu_{i-1})$ then ν_i is an entry to μ . Since ν is a cycle, then $v \geq s(\nu_i)$. If we suppose $s(\nu_i) \in H$ then because H is hereditary, this would imply $v \in H$, which cannot be true. So $s(\nu_i) \notin H$ and thus, $\nu_i \in s^{-1}(E^0 \setminus H) = (E \setminus H)^1$, proving the first direction.

To see the converse, we now take a vertex v and a cycle $\mu = \mu_1 \dots \mu_n \in E^*$ with $s(\mu) = v$. To prove condition (K), we must find an appropriate path ν . Let's first consider the collection $H = \{w \mid v \not\geq w\}$.

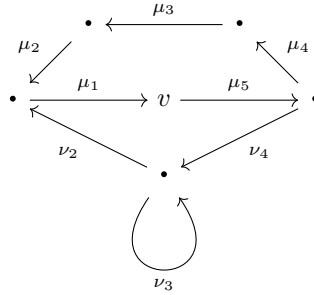
We claim that H is hereditary and saturated. Fix some $w \in H$ and suppose we take some $z \in E^0$ with $z \geq w$. If $z \notin H$ then we would have $v \geq z \geq w$ which contradicts the assumption that $z \notin H$. Thus, $z \in H$ so H is hereditary.

We will now see that H is saturated. Take some vertex w such that $r^{-1}(w) \neq \emptyset$ and $\{s(e) \mid r(e) = w\} \subseteq H$. If $w \notin H$ then we would be able to find a path α with $s(\alpha) = v$ and $r(\alpha) = w$. Then $s(\alpha_1) \in H$ by the assumption on w . However, because $v \geq s(\alpha_1)$ and H has already been shown to

be hereditary, then $v \in H$. This clearly cannot be true, so w must be an element of H_μ . Thus, H_μ is saturated.

We have already shown in Lemma ?? that H must be hereditary and saturated. Because the cycle μ lies in $E \setminus H$, then it must have an entry e in $E \setminus H$ by the proposition assumption. By choosing an appropriate index i , we may assume $r(e) = r(\mu_i)$ so that $s(e) \geq r(e) = r(\mu_i) \geq v$. Then $s(e) \notin H$. Since $v \geq s(e)$ in $E \setminus H$, we can find a path β with $s(\beta) = v$ and $r(\beta) = s(e)$. By setting $\nu = \mu_i \dots \mu_{i-1} e \beta$ we have the desired distinct return path, and so E satisfies condition (K). \square

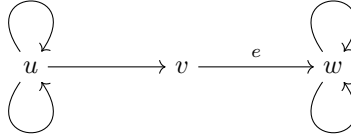
Example 4.20. To illuminate the forward direction of the proof, consider the graph $E \setminus H$ in Graph 21. Suppose we are given $\mu = \mu_1 \mu_2 \mu_3 \mu_4 \mu_5$ and we find the path $\nu = \mu_1 \nu_2 \nu_3 \nu_4 \mu_5$. In this case, we would choose i to be 2.



GRAPH 21

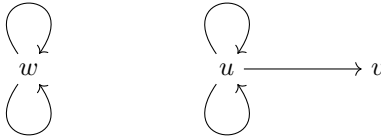
Definition 4.21. Given a saturated, hereditary set H , define $E_H = (H, r^{-1}(H), r, s)$, whose vertex set is the complement of the vertex set belonging to $E \setminus H$.

Example 4.22. We now look at Graph 22.



GRAPH 22

Taking $H = \{u, v\}$ we have the subgraphs $E \setminus H$ and E_H , shown in Graph 23.



GRAPH 23. On the left we have $E \setminus H$ and on the right we have E_H

Note that the edge e is not included in either subgraph. This observation tells us that we cannot recover the graph E from $E \setminus H$ and E_H . We will now work towards a complete description of the (closed) ideals of $C^*(E)$.

Definition 4.23. Given a saturated, hereditary set H , let I_H be the ideal in $C^*(E)$ generated by the set $\{p_v \mid v \in H\}$.

While we may define I_H for any collection H , we will see in Theorem 4.28 why we restrict our attention to saturated, hereditary sets.

Lemma 4.24. *Let E be a row-finite graph which satisfies condition (K). Given an ideal I of $C^*(E)$, $I = I_{H_I}$. Moreover, $C^*(E \setminus H) \cong C^*(E)/I_H$.*

Proof. Set $H = H_I$ to avoid requiring double subscripts throughout the proof. Lemma 4.14 tells us that H is saturated and hereditary, so I_H is consistent with our definition above. First note that if $p_v \in I_H$ then $v \in H$ and so $p_v \in I$. Since the generators of I_H are in I , we have the inclusion $I_H \subseteq I$. Now consider the quotient maps

$$\begin{aligned} q^I &: C^*(E) \rightarrow C^*(E)/I, \\ q^{I_H} &: C^*(E) \rightarrow C^*(E)/I_H, \text{ and} \\ q^{I/I_H} &: C^*(E)/I_H \rightarrow C^*(E)/I = (C^*(E)/I_H)/(I/I_H), \end{aligned}$$

where $q^I = q^{I/I_H} \circ q^{I_H}$. Note that the only projections q^I or q^{I_H} send to zero are the p_v with $v \in H$. Since the two quotient maps kill the same projections, they also kill the same partial isometries s_e , since $s_e^* s_e = p_{s(e)}$.

It is not hard to check that $\{q^I(s_e), q^I(p_v)\}$ is a CK $(E \setminus H)$ -family which generates $C^*(E)/I$ and similarly, $\{q^{I_H}(s_e), q^{I_H}(p_v)\}$ is a CK $(E \setminus H)$ -family which generates $C^*(E)/I_H$. By universality, we have the homomorphisms

$$\begin{aligned} \pi &: C^*(E \setminus H) \rightarrow C^*(E)/I_H \\ \rho &: C^*(E \setminus H) \rightarrow C^*(E)/I. \end{aligned}$$

We may consider the composition $q^{I/I_H} \circ \pi : C^*(E \setminus H) \rightarrow C^*(E)/I$ which agrees with ρ on the generators of $C^*(E \setminus H)$. Hence, $\rho = q^{I/I_H} \circ \pi$.

Since the graph E satisfies condition (K), every cycle in $E \setminus H$ has an entry. We may apply the CK-Uniqueness Theorem to conclude that ρ is injective and π is surjective, we have that q^{I/I_H} is injective as well. Thus, $I = I_H$. Moreover, the CK-Uniqueness Theorem says that the map ρ from $C^*(E \setminus H)$ to $C^*(E)/I_H$ is an isomorphism. \square

Lemma 4.25. *Let E be a row-finite and let H be a saturated and hereditary subset of E^0 . Then $H = \{v \mid p_v \in I_H\}$.*

Proof. The inclusion $H \subseteq \{v \mid p_v \in I_H\}$ is immediate. To see the converse, consider the universal $(E \setminus H)$ -family $\{t, q\}$ which generates $C^*(E \setminus H)$. We will extend this to a CK E -family by defining $t_e = 0$ if $s(e) \in H$ and $q_v = 0$ if $v \in H$.

To confirm that this is indeed a CK E -family, if $s(e) \in H$ then $t_e^* t_e = 0 = q_{s(e)}$. For a vertex $v \in H$ since H is hereditary, then if $r(e) = v$ we have $s(e) \in H$ and so $t_e = 0$. Thus, $q_v = 0 = \sum_{r(e)=v} t_e t_e^*$ as desired.

Now consider a vertex $v \in E \setminus H$. If v is not a source in $E \setminus H$ then

$$\begin{aligned} q_v &= \sum_{e \in E \setminus H, r(e)=v} t_e t_e^* \\ &= \sum_{e \in E \setminus H, r(e)=v} t_e t_e^* + \sum_{r(e) \in H, r(e)=v} \underbrace{t_e t_e^*}_{=0} \\ &= \sum_{e \in E, r(e)=v} t_e t_e^*. \end{aligned}$$

On the other hand, if v is a source in $E \setminus H$ then for an edge e , if $r(e) = v$ then $s(e) \in H$ as well. Thus, $\{s(e) \mid r(e) = v\} \subseteq H$ and so by saturation $v \in H$. This is a contradiction, and so any sources v in $E \setminus H$ are still sources in E . So indeed, our collection $\{t, q\}$ is a CK E -family.

Consider the homomorphism $\pi_{t,q} : C^*(E) \rightarrow C^*(E \setminus H)$. For a vertex $v \in H$, we have $\pi_{t,q}(p_v) = 0$. If $v \notin H$ then $\pi_{t,q}(p_v) = q_v \neq 0$ and so $p_v \notin I_H$. Hence, $H = \{v \mid p_v \in I_H\}$. \square

Lemma 4.26. *Let E be a row-finite graph and X be a hereditary subset of E^0 . Set ΣX to be the smallest saturated set containing X . Then ΣX is also hereditary.*

Proof. Suppose to the contrary that ΣX is not hereditary. There would exist some $v \notin \Sigma X$ and $w \in \Sigma X$ with $w \leq v$. By truncating the path between v and w and possibly modifying our choice of vertices, we may assume without loss of generality that there exists an edge f from v to w .

We claim that $\Sigma X \setminus \{w\}$ is a smaller saturated set containing X . First, we note that if $w \in X$ then since X is hereditary, $v \in X \subseteq \Sigma X$, contradicting our assumption that $v \notin \Sigma X$. Thus, $w \notin X$, so $\Sigma X \setminus \{w\}$ contains X .

We now see that $\Sigma X \setminus \{w\}$ is saturated. For any vertex u which is different from w , if $\{s(e) \mid r(e) = u\} \subseteq \Sigma X \setminus \{w\} \subseteq \Sigma X$ then because ΣX is saturated, we have $u \in \Sigma X \setminus \{w\}$. Now, since $v \in \{s(e) \mid r(e) = w\}$ then $\{s(e) \mid r(e) = w\} \not\subseteq \Sigma X \setminus \{w\}$ and so removing the vertex w does not affect the saturation of ΣX . Thus, the assumption that ΣX is not hereditary allows us to build a smaller saturated set containing X , contradicting the minimality of ΣX . Hence, ΣX must be hereditary. \square

Lemma 4.27. *Let E be a row-finite graph, X be a hereditary subset of E^0 , and set ΣX to be the smallest saturated set containing X . Then there is an isomorphism of $C^*(E_X)$ onto the corner $p_X I_{\Sigma X} p_X$, where the projection $p_X = \sum_{v \in \Sigma X} p_v$ was defined in Lemma 3.18.*

Proof. By Lemma 4.26, the set ΣX is hereditary and so $I_{\Sigma X}$ makes sense. We claim that $I_{\Sigma X} = \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \in \Sigma X\}$. First fix paths μ, ν such that $s(\mu), s(\nu) \in \Sigma X$ and consider paths $\alpha, \beta \in E^*$. Because we have

$$(s_\mu s_\nu^*)(s_\alpha s_\beta^*) = \begin{cases} s_{\mu\alpha'} s_{\beta}^* & \text{if } \alpha = \nu\alpha' \\ s_\mu s_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise,} \end{cases}$$

it follows that $s_\mu s_\nu^* s_\alpha s_\beta^* = s_\sigma s_\tau^*$ where $s(\mu) \leq s(\sigma) = s(\tau)$. Since ΣX is hereditary, $s(\sigma) \in \Sigma X$. Thus, $\overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \in \Sigma X\}$ is an ideal of $C^*(E)$. Moreover, it contains the generators of $I_{\Sigma X}$ so $I_{\Sigma X} \subseteq \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \in \Sigma X\}$.

Finally, if $s(\mu) = s(\nu) \in \Sigma X$ then $s_\mu s_\nu^* = s_\mu p_{s(\mu)} s_\nu^* \in I_{\Sigma X}$ so the equality holds. By continuity and linearity of the projections p_X , we have

$$p_X I_{\Sigma X} p_X = \overline{\text{span}}\{s_\mu s_\nu^* \mid s(\mu) = s(\nu) \in \Sigma X, \quad r(\mu) \in X, \quad r(\nu) \in X\}.$$

Then $\{s_e \mid r(e) \in \Sigma X\} \cup \{p_v \mid v \in \Sigma X\}$ is a CK E_X -family in $p_X I_{\Sigma X} p_X$ which generates $p_X I_{\Sigma X} p_X$. Since every cycle in E_X has an entry in E , by hereditariness of X , it also has an entry in E_X . Thus, we may apply the CK-Uniqueness Theorem to see that $p_X I_{\Sigma X} p_X$ is isomorphic to $C^*(E_X)$. \square

We now compile the above lemmas to present the following theorem.

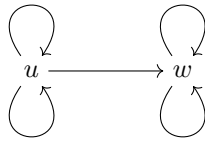
Theorem 4.28. *Suppose E is a row-finite graph which satisfies condition (K). We have the correspondence*

$$\begin{aligned} \{\text{saturated, hereditary subsets of } E^0\} &\leftrightarrow \{\text{closed ideals in } C^*(E)\} \\ H &\mapsto I_H \\ H_I &\leftarrow I. \end{aligned}$$

Moreover, $C^*(E)/I_H$ is isomorphic to $C^*(E \setminus H)$ and $C^*(E_H)$ is isomorphic to the corner $p_H I_H p_H$.

Proof. Surjectivity of the map is given by Lemma 4.24 and injectivity is given by Lemma 4.25. Thus, we do indeed have the desired bijection and $C^*(E)/I_H \cong C^*(E \setminus H)$. Finally, since H is saturated and hereditary, then $\Sigma H = H$ and so by Lemma 4.27, $C^*(E_H)$ is isomorphic to the corner $p_H I_H p_H$. \square

Example 4.29. Let's apply this theorem to Graph 24.



GRAPH 24

The hereditary sets are \emptyset , $\{w, u\}$ and $\{u\}$. We see that each of these sets are saturated. Thus, Theorem 4.28 tells us that the ideals in $C^*(E)$ are $I_\emptyset = \{0\}$, $I_{\{u, w\}} = C^*(E)$ and $I_{\{u\}}$. Since $E_{\{u\}}$ consists only of the vertex u and the two loops at u , then $C^*(E_{\{u\}}) = \mathcal{O}_2$. Thus, $p_{\{u\}} I_{\{u\}} p_{\{u\}}$ is isomorphic to \mathcal{O}_2 . Moreover, because the graph for $E \setminus H$ is again a single vertex, w , with two loops then $C^*(E \setminus H) = \mathcal{O}_2 \cong C^*(E)/I_{\{u\}}$.

We now conclude this section with a theorem which tells us precisely when $C^*(E)$ is simple.

Theorem 4.30. *For a row-finite graph E , $C^*(E)$ is simple if and only if every cycle in E has an entry and E is cofinal.*

Proof. We have already shown the ‘only if’ direction in Theorem 4.4 and are only left to prove the ‘if’ direction. Suppose $C^*(E)$ is simple. We will first prove that E is cofinal. Take an arbitrary path $\mu \in E^*$. By Theorem ?? we have H_μ is saturated and hereditary. If H_μ were non-trivial then I_{H_μ} would be a proper ideal, contradicting the assumption that $C^*(E)$ is simple. Since $r(\mu) \notin H_\mu$ then we know H_μ cannot be all of E and so $H_\mu = \emptyset$. Thus, E is cofinal.

We will next see that every cycle has an entry. Suppose this were not the case, and find a cycle $\mu \in E^*$ with no entry. Define X to be the hereditary set $\{s(\mu_i) \mid 1 \leq i \leq |\mu|\} \neq \emptyset$. As in Lemma 4.27, let ΣX be the smallest saturated set containing X . If ΣX were non-trivial, then $I_{\Sigma X}$ is a proper ideal. Since $C^*(E)$ is simple, then $\Sigma X = E^0$ and so $C^*(E_X) \cong p_X I_{\Sigma X} p_X = p_X C^*(E) p_X$ by Lemma 4.27. However, it’s clear that $E_X = X$ and so the discussion of Graph 7 tells us that $C^*(E_X) \cong \mathcal{C}(\mathbb{T}, M_{|\mu|}(\mathbb{C})) \cong M_{|\mu|}(\mathcal{C}(\mathbb{T}))$.

Let J be an arbitrary proper ideal in $p_X C^*(E) p_X \cong C^*(E_X) \cong \mathcal{C}(\mathbb{T}, M_{|\mu|}(\mathbb{C}))$. Since we may note that the collection of functions in $\mathcal{C}(\mathbb{T}, M_{|\mu|}(\mathbb{C}))$ which vanish at 1 form a proper ideal of the space, such a proper ideal J exists. Consider the set

$$C^*(E)JC^*(E) = \overline{\text{span}}\{ajb \mid a, b \in C^*(E), j \in J\},$$

which is a non-zero ideal in $C^*(E)$. Finally, since $C^*(E)$ is simple, $C^*(E)JC^*(E) = C^*(E)$, and thus

$$p_X C^*(E) p_X = p_X (C^*(E)JC^*(E)) p_X = p_X C^*(E) (p_X J p_X) C^*(E) p_X = J.$$

However we assumed J to be a proper ideal of the left hand side, proving that our assumption of the cycle μ having no cycle was incorrect. Thus, every cycle in E has an entry. □

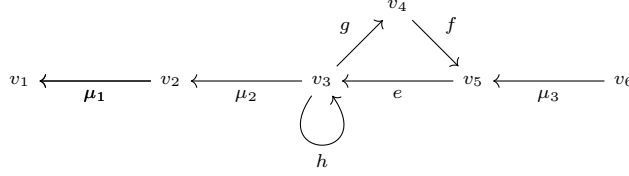
We have now completely classified the graphs for which $C^*(E)$ is simple, as well as investigated the ideals of $C^*(E)$, when they exist.

5. THE ABELIAN CORE

In this section, we will investigate the normal elements of $C^*(E)$ which generate the abelian core of a graph algebra. We will spend much of this section giving the appropriate background in order to present a new uniqueness theorem whose proof considers a particular representation on the set of all essentially aperiodic trails. Szymański first proved this result in [12]. It was revisited by Nagy and Reznikoff in [9] in 2012, and it is their proof that we present in this section.

Definition 5.1. We call an infinite path $\mu = \mu_1 \mu_2 \dots$ **periodic** if there exists positive integers j and k such that $\mu_{n+k} = \mu_n$ for all $n \geq j$. For such a periodic path μ , we can find a closed path ν of length k such that $\mu = \mu_1 \dots \mu_{j-1} \nu \nu \nu \dots$ where j and k are chosen to be minimal and we call ν the **period** of μ .

Example 5.2. Given Graph 25, the path $\mu = \mu_1\mu_2(hefg)(hefg)(hefg)\dots$ is periodic since for $n \geq 3$ we have $\mu_{n+4} = \mu_n$. In this case, minimality of j and k force the period to be $\nu = hefg$.



GRAPH 25

Definition 5.3. We call the path $\tau = \tau_1\tau_2\dots$ an **essentially aperiodic trail** if any of the following hold:

- (1) τ is finite and $s(\tau)$ is a source,
- (2) τ is infinite and periodic but its period has no entries (making its period an entryless cycle),
- (3) τ is infinite and not periodic.

We loosen this definition a bit and call τ a **trail** if it either satisfies (1) or is an infinite path. The set of all trails was denoted $E^{\leq\infty}$ in Section 4. Essentially aperiodic trails of form (1) or (2) are called **discrete** and those of form (3) are called **continuous**. Note that if τ is an essentially aperiodic trail and α is a path with $s(\alpha) = r(\tau)$ then $\alpha\tau$ is also an essentially aperiodic trail. Let \mathcal{T} be the collection of all essentially aperiodic trails.

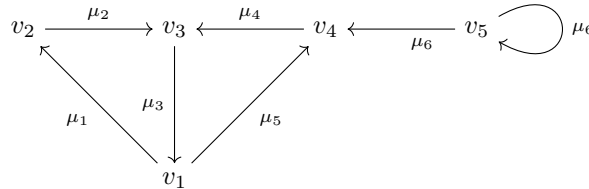
Example 5.4. In Graph 25 above, we see that all finite paths which begin at v_6 will be essentially aperiodic trails, while any other finite paths will not be essentially aperiodic. Since the graph has no entryless cycles, we cannot construct an essentially aperiodic trail of form (2).

Despite the fact that the graph is finite, we may still construct an infinite path τ which is not periodic. Indeed, if we set $\nu = efg$ then one such construction is $\tau = \nu h \nu h h \nu h h h \dots$, which we may confirm is not periodic.

Definition 5.5. We call a vertex $w \in E^0$ a **trap** if

- (a) w is a source,
- (b) w lies on an entryless loop, or
- (c) there exists two closed paths μ, ν with $s(\mu) = s(\nu) = r(\mu) = r(\nu) = w$ such that $\mu_1 \neq \nu_1$.

Example 5.6. Returning to Graph 25, we see that v_6 is a trap because it is a source. The vertex v_3 is also a trap since $\mu = h$ and $\nu = efg$ are closed paths where $\mu_1 = h \neq e = \nu_1$. Now consider Graph 26.



GRAPH 26

For this graph, v_5 lies on the entryless loop μ_6 , making v_5 a trap. We may construct the two closed paths $\mu = \mu_4\mu_5\mu_3$ and $\nu = \mu_2\mu_1\mu_3$ which both start and end at v_3 where the first edge in the two paths are not equal. Thus, v_3 is also a trap. We may confirm that v_1, v_2 and v_4 are not traps.

Lemma 5.7. *Suppose E is a row-finite graph. For every vertex $v \in E^0$, there exists an essentially aperiodic trail τ with $v = r(\tau)$.*

Proof. We split the proof into two cases. First suppose there exists a path α with $r(\alpha) = v$ such that $s(\alpha)$ is a trap. There are now three subcases describing the nature of $s(\alpha)$. First, if $s(\alpha)$ is a source, we may set $\tau = \alpha$. Second, if $s(\alpha)$ sits on an entryless loop λ , then we may set $\tau = \alpha\lambda\lambda\lambda\dots$ to be a periodic trail. Finally, suppose there are two paths λ, τ as in the third case of a trap. Since we may replace both λ and τ with a multiple of themselves, we may assume without loss of generality that $|\lambda| = |\tau|$. Now let $\tau = \alpha\lambda\tau\lambda\tau\lambda\tau\lambda\tau\dots$. We claim that this path is not periodic. Indeed, if it had period k and we were to label the segment $\lambda\tau\lambda\tau\dots\tau$ where τ is repeated k times by edges $e_1e_2\dots e_{(k+1)|\lambda|}$, then $\lambda_1 = e_1 = e_{1+|\lambda|k} = \tau_1$, which contradicts our choice of λ and τ . Thus, τ cannot be periodic. This concludes the case where there exists a path α with $r(\alpha) = v$ and $s(\alpha)$ a trap.

Now suppose such a path does not exist. We will construct a sequence (τ_n) of paths with $r(\tau_1) = v$ such that

$$s(\tau_{n-1}) = r(\tau_n) \quad \text{and} \quad [\tau_n] \not\subseteq [\tau_1 \dots \tau_{n-1}],$$

where $[\tau]$ is the collection of vertices visited by τ . If we can show this, then $\tau = \tau_1\tau_2\tau_3\dots$ will visit infinitely many vertices, making it an essentially aperiodic trail. Let τ_1 be an arbitrary edge with $r(\tau_1) = v$, and now suppose we have constructed the first N paths. We will now find an appropriate τ_{N+1} .

Write the path $\tau_1 \dots \tau_N$ as edges $e_1e_2\dots e_k$. Since $s(e_k)$ is not a trap, there exists an edge e_{k+1} with $r(e_{k+1}) = s(e_k)$. If $s(e_{k+1}) \notin [\tau_1 \dots \tau_N]$, then we may let $\tau_{N+1} = e_{k+1}$. However, if $s(e_{k+1}) \in [\tau_1 \dots \tau_N]$ then there is a closed path $\lambda = e_{k+1}e_pe_{p+1}\dots e_k$ for some $p \in \{1, \dots, k+1\}$. Since no vertex in $[\lambda]$ is a trap, then λ has an entry, call it e_{k+2} .

We now claim that $s(e_{k+2}) \notin [\tau_1 \dots \tau_N]$. Suppose this were not the case; we would be able to find a closed path ν with $r(\nu) = s(\nu) = s(e_{k+2})$ which is a segment of $\tau_1 \dots \tau_N$. We've now found two appropriate paths with $r(\lambda) = s(\lambda) = r(\nu) = s(\nu) = s(e_{k+2})$ proving that $s(e_{k+2})$ is a trap. This is a contradiction, and so $s(e_{k+2}) \notin [\tau_1 \dots \tau_N]$. We may set $\tau_{N+1} = e_{k+1}e_pe_{p+1}\dots e_{k+2}$.

This now proves that in either case, there exists an essentially aperiodic trail τ with $r(\tau) = v$. \square

5.1. The diagonal. We now introduce some terminology which will be used in the following results. We will show in Section 5.2 the relation between the diagonal and the abelian core. For now, the examples in this section will justify the terminology.

Definition 5.8. For an arbitrary CK E -family $\{S, P\}$, we let $G^\Delta(S, P) = \{S_\mu S_\mu^* \mid \mu \in E^*\}$ and define the **diagonal** to be

$$\Delta(S, P) = C^*(G^\Delta(S, P)) = \overline{\text{span}}\{S_\mu S_\mu^* \mid \mu \in E^*\}.$$

Now, if $\mu = \nu\mu'$ then $(S_\nu S_\nu^*)(S_\mu S_\mu^*) = S_{\nu\mu'} S_\mu = S_\mu S_\mu^* = (S_\mu S_\mu^*)(S_\nu S_\nu^*)$. If $\nu = \mu\nu'$ then $(S_\nu S_\nu^*)(S_\mu S_\mu^*) = S_\nu S_{\mu\nu'}^* = S_\nu S_\nu^* = (S_\mu S_\mu^*)(S_\nu S_\nu^*)$. Finally, if μ does not extend ν and ν does not extend μ then $(S_\nu S_\nu^*)(S_\mu S_\mu^*) = 0 = (S_\mu S_\mu^*)(S_\nu S_\nu^*)$.

Thus, the generators of $\Delta(S, P)$ are commuting projections. If $\{s, p\}$ is the universal CK E -family which generates $C^*(E)$, then we write $G_E^\Delta = G^\Delta(s, p)$ and $\Delta(E) = \Delta(\{s, p\})$.

Note that if $\{T, Q\}$ is an arbitrary CK E -family in some C^* -algebra \mathfrak{B} then $\pi_{T, Q} : C^*(E) \rightarrow \mathfrak{B}$ maps G_E^Δ onto $G^\Delta(\{T, Q\})$. Thus, $\pi_{T, Q}$ also maps $\Delta(E)$ onto $\Delta(\{T, Q\})$, so we say that $\Delta(E)$ is universal in this sense. Let's take a look at a few examples to get a handle on these new terms.

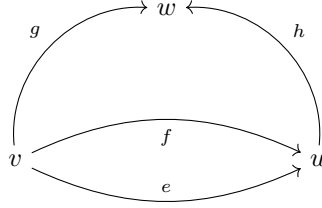
Example 5.9. Consider the graph E to be the single edge shown in Graph 27.

$$v \xrightarrow{e} w$$

GRAPH 27

Then we have $E^* = \{v, w, e\}$ and so $G_E^\Delta = \{p_v, p_w, s_e s_e^* = p_w\} = \{p_v, p_w\}$. Now, the identity element in $C^*(E)$ is $I = p_v + p_w$ which is an element of $\Delta(E)$, but is not contained in G_E^Δ .

Example 5.10. Recall Graph 3, shown again below.



In Section 2.2, we found a faithful representation of the universal CK E -family $\{s, p\}$ on \mathbb{C}^6 to be

$$\begin{aligned} s_e &= E_{21} & s_f &= E_{31} & s_g &= E_{41} & s_h &= E_{52} + E_{63} \\ p_v &= E_{11} & p_u &= E_{22} + E_{33} & p_w &= E_{44} + E_{55} + E_{66}. \end{aligned}$$

Then $s_{he} s_{he}^* = E_{55}$ and $s_{hf} s_{hf}^* = E_{66}$ and since $E^* = \{v, u, w, e, f, g, h, he, hf\}$, we have $G_E^\Delta = \{E_{11}, E_{22} + E_{33}, E_{44} + E_{55} + E_{66}, E_{22}, E_{33}, E_{44}, E_{55} + E_{66}, E_{55}, E_{66}\}$. Thus, $\Delta(E) = \text{span}\{E_{ii} \mid 1 \leq i \leq 6\}$.

Definition 5.11. We define

$$\Delta^{\leq k}(E) = \overline{\text{span}\{s_\alpha s_\alpha^* \mid \alpha \in E^{\leq k}\}}$$

where $E^{\leq k}$ was defined in Section 3.1. For $\alpha \in E^{\leq k}$ if $s(\alpha)$ is a source, then $s_\alpha s_\alpha^* \in \Delta^{\leq k+1}(E)$. If $s(\alpha)$ is not a source, then $s_\alpha s_\alpha^* = \sum_{r(e)=s(\alpha)} s_{\alpha e} s_{\alpha e}^* \in \Delta^{\leq k+1}(E)$.

Thus, $\Delta^{\leq k}(E) \subseteq \Delta^{\leq k+1}(E)$ and $\Delta(E) = \bigcup_{k \geq 0} \Delta^{\leq k}(E)$.

Proposition 5.12. *If E has no sources, then for a $*$ -homomorphism $\Phi : \Delta(E) \rightarrow \mathfrak{A}$, the following are equivalent:*

- (1) Φ is injective
- (2) $\Phi(s_\mu s_\mu^*) \neq 0$ for all $\mu \in E^*$
- (3) $\Phi(p_v) \neq 0$ for all $v \in E^0$.

Proof. (1) \Rightarrow (3) This is clear.

(3) \Rightarrow (2) We may write $\Phi(s_\mu s_\mu^*) = \Phi(s_\mu)\Phi(s_\mu^* s_\mu)\Phi(s_\mu^*) = \Phi(s_\mu)\Phi(p_{s(\mu)})\Phi(s_\mu^*)$. Thus, if $\Phi(s_\mu) = 0$ then $\Phi(p_{s(\mu)}) = 0$ so $\Phi(s_\mu s_\mu^*) = 0$.

(2) \Rightarrow (1) For some $\alpha, \beta \in E^{\leq k}$ with $\alpha \neq \beta$, $s_\alpha^* s_\beta = 0$. Thus, $\Delta^{\leq k}(E) = \oplus_{\alpha \in E^{\leq k}} \mathbb{C} s_\alpha s_\alpha^*$. Since $\Phi(s_\alpha s_\alpha^*) \neq 0$, then Φ is injective on each $\Delta^{\leq k}(E)$ for all $k \geq 0$. Thus, Φ is injective. □

Note the similarities between this proof and that of Lemma 3.10.

The proof of our final uniqueness theorem involves considering a particular faithful representation of $C^*(E)$ on $\ell^2(\mathcal{T} \times \mathbb{Z})$. In the following section, we will see that this particular representation may be intertwined with a conditional expectation to explain the relation between the diagonal and the abelian core.

Theorem 5.13. *Suppose E is a row-finite graph with no sources and let \mathcal{T} be the set of all essentially aperiodic trails in E . Consider the standard orthonormal basis $(\xi_\tau^n)_{\tau \in \mathcal{T}, n \in \mathbb{Z}}$ for $\ell^2(\mathcal{T} \times \mathbb{Z})$. Then there exists a unique CK E -family $\{S, P\} \subseteq \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ such that for every path $\alpha \in E^*$,*

$$S_\alpha \xi_\tau^n = \begin{cases} \xi_{\alpha\tau}^{n+|\alpha|} & \text{if } r(\tau) = s(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, the $*$ -homomorphism $\pi_{S,P} : C^*(E) \rightarrow \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ is injective.

Proof. Fix some path $\alpha \in E^*$ and consider the set $\mathcal{M} = \{\xi_\tau^n \mid \tau \in \mathcal{T}, n \in \mathbb{Z}\}$ of all basis elements. Let $S_\alpha^0 : \mathcal{M} \rightarrow \mathcal{M} \cup \{0\}$ be defined by

$$S_\alpha^0(\xi_\tau^n) = \begin{cases} \xi_{\alpha\tau}^{n+|\alpha|} & \text{if } r(\tau) = s(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}_\alpha^0 = \{\xi_\tau^n \in \mathcal{M} \mid n \in \mathbb{Z}, \tau \in \mathcal{T}, s(\alpha) \neq r(\tau)\}$ so that $S_\alpha^0|_{\mathcal{M}_\alpha^0} \equiv 0$. If we take distinct ξ_τ^n and ξ_γ^m in $\mathcal{M} \setminus \mathcal{M}_\alpha^0$ then either $n \neq m$ or $\tau \neq \gamma$. In either case, $S_\alpha^0(\xi_\tau^n) = \xi_{\alpha\tau}^{n+|\alpha|} \neq \xi_{\alpha\gamma}^{m+|\alpha|} = S_\alpha^0(\xi_\gamma^m)$. Thus, $S_\alpha^0|_{\mathcal{M} \setminus \mathcal{M}_\alpha^0}$ is injective. Note that this implies that S_α^0 is also injective on $\overline{\text{span}}(\mathcal{M} \setminus \mathcal{M}_\alpha^0)$.

Thus, we may extend S_α^0 to a partial isometry S_α in $\mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ with $\ker(S_\alpha) = \overline{\text{span}}(\mathcal{M}_\alpha^0)$ and $\text{ran } S_\alpha = \overline{\text{span}}\{\xi_\tau^n \mid n \in \mathbb{Z}, \tau = \alpha\tau'\}$. It is not hard to confirm that

$$S_\alpha^* \xi_\tau^n = \begin{cases} \xi_{\tau'}^{n-|\alpha|} & \text{if } \tau = \alpha\tau' \\ 0 & \text{otherwise.} \end{cases}$$

We will now check that the collection $\{P_v = S_v, S_e\}$ is a CK E -family. For a fixed $v \in E^0$, if $\xi_\tau^n \in \mathcal{M} \setminus \mathcal{M}_v^0$ then

$$P_v \xi_\tau^n = S_v \xi_\tau^n = \xi_{v\tau}^{n+0} = \xi_\tau^n = P_v^* \xi_\tau^n.$$

Thus, P_v is a projection onto $\overline{\text{span}}(\mathcal{M} \setminus \mathcal{M}_v^0) = \overline{\text{span}}\{\xi_\tau^n \mid r(\tau) = v\}$. We may verify that $S_e S_e^* = P_{s(e)}$ by checking this equality at each basis element ξ_τ^n . Finally, we fix $v \in E^0$ and some orthonormal basis element ξ_τ^n . If $r(\tau) \neq v$ then we have $P_v \xi_\tau^n = 0 = \sum_{r(e)=v} S_e S_e^* \xi_\tau^n$. If $r(\tau) = v$ then we may write $\tau = e\tau'$ and so

$$\left(P_v - \sum_{r(e)=v} S_e S_e^* \right) \xi_\tau^n = \xi_\tau^n - \sum_{r(e)=v} \xi_{e\tau'}^n,$$

where our assumption that E is row-finite tells us that the two sums are finite. There is precisely one edge e where $\tau = e\tau'$, so independent of whether $r(\tau) = v$ or not, we have $\left(P_v - \sum_{r(e)=v} S_e S_e^* \right) \xi_\tau^n = 0$ for all ξ_τ^n . Thus, $P_v = \sum_{r(e)=v} S_e S_e^*$. This proves that $\{P_v, S_e\}$ is indeed a CK E -family, and so it makes sense to talk about the *-homomorphism $\pi_{S,P} : C^*(E) \rightarrow \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$. By again checking on the orthonormal basis elements, it is not hard to confirm that for a path $\alpha = \alpha_1 \dots \alpha_n$, the operator S_α is equal to the composition $S_{\alpha_1} S_{\alpha_2} \dots S_{\alpha_n}$, and so it is only left to show that the map $\pi_{S,P}$ is injective.

By Lemma 5.7, for every vertex $v \in E^0$, there exists an essentially aperiodic trail $\tau \in \mathcal{T}$ with $v = r(\tau)$. Thus, $P_v \xi_\tau^n = \xi_\tau^n$ and so $P_v \neq 0$. Consider the unitary operator $U_z \in \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ defined by $U_z \xi_\tau^n = z^n \xi_\tau^n$ with adjoint $U_z^* \xi_\tau^n = z^{-n} \xi_\tau^n$. We can then define the action $\beta : \mathbb{T} \rightarrow \text{Aut } \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ to act on $X \in \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ by $\beta_z(X) = U_z X U_z^*$. For any fixed $\alpha \in E^*$ and $z \in \mathbb{T}$, we may check that $\beta_z(S_\alpha) = U_z S_\alpha U_z^* = z^{|\alpha|} S_\alpha$.

More precisely, $\beta_z(P_v) = U_z P_v U_z^* = P_v$ and $\beta_z(S_e) = U_z S_e U_z^* = z S_e$. Thus, the Gauge-Invariant Uniqueness Theorem (Theorem 3.11) tells us that $\pi_{S,P}$ is injective. \square

We rename this unique *-homomorphism to be π_{ap} .

Definition 5.14. For an essentially aperiodic trail τ and a path $\alpha \in E^*$, define the projections $R_\alpha, Q_\tau \in \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ by

$$Q_\tau = \text{proj}_{\overline{\text{span}}\{\xi_\tau^n \mid n \in \mathbb{Z}\}} \quad \text{and} \quad R_\alpha = \text{proj}_{\overline{\text{span}}\{\xi_\tau^n \mid \tau \in \mathcal{T}, n \in \mathbb{Z}, \tau = \alpha\tau'\}}.$$

Note that in the proof of Theorem 5.13, we found that $\overline{\text{span}}\{\xi_\tau^n \mid \tau \in \mathcal{T}, n \in \mathbb{Z}, \tau = \alpha\tau'\} = \text{ran}(S_\alpha)$ and so because S_α is a partial isometry, $R_\alpha = \text{proj}_{\text{ran } S_\alpha} = S_\alpha S_\alpha^*$.

Definition 5.15. We define the **tail of length** n for a trail $\tau = \tau_1\tau_2\cdots \in \mathcal{T}$ to be

$$\tau_{(n)} = \begin{cases} r(\tau) & \text{if } n = 0 \\ \tau_1\tau_2\cdots\tau_n & \text{if } n > 0 \text{ and } \tau \text{ is either infinite or finite with } |\tau| > n \\ \tau & \text{if } \tau \text{ is finite, with } |\tau| \leq n. \end{cases}$$

Note that for any trail τ and any positive integer n , the tail $\tau_{(n)}$ is a path. Thus, $R_{\tau_{(n)}}$ is well-defined.

Proposition 5.16. *For a row-finite graph E , an essentially aperiodic trail $\tau \in \mathcal{T}$ with tails $\tau_{(n)}$, and the operators $R_{\tau_{(n)}}$ and Q_τ defined above, we have*

$$\text{SOT} - \lim_n R_{\tau_{(n)}} = Q_\tau.$$

Proof. Fix any basis element $\xi_\gamma^m \in \mathcal{M}$ and note that we may write $\gamma = \tau_{(n)}\gamma'$ if and only if $\gamma = \tau$. Thus,

$$\lim_n S_{\tau_{(n)}} S_{\tau_{(n)}}^* \xi_\gamma^m = \lim_n R_{\tau_{(n)}} \xi_\gamma^m = \lim_n \begin{cases} \xi_\gamma^m & \text{if } \gamma = \tau \\ 0 & \text{otherwise} \end{cases} = Q_\tau \xi_\gamma^m.$$

Since each $R_{\tau_{(n)}}$ is a projection, it is bounded and therefore continuous. Thus, for any $x \in \ell^2(\mathcal{T} \times \mathbb{Z})$ we have that $\lim_n R_{\tau_{(n)}} x = Q_\tau x$, giving $\text{SOT} - \lim_n R_{\tau_{(n)}} = Q_\tau$. \square

We now notice that for $\alpha \in E^*$, if $\gamma = \tau = \alpha\gamma'$ then

$$\begin{aligned} Q_\tau R_\alpha \xi_\gamma^n &= Q_\tau \xi_\gamma^n = \xi_\tau^n & \text{and} \\ R_\alpha Q_\tau \xi_\gamma^n &= R_\alpha \xi_\tau^n = \xi_\tau^n. \end{aligned}$$

If this assumption does not hold, then

$$Q_\tau R_\alpha \xi_\gamma^n = 0 \quad \text{and} \quad R_\alpha Q_\tau \xi_\gamma^n = 0.$$

Thus, $Q_\tau R_\alpha = R_\alpha Q_\tau$.

Recall that $\pi_{ap} : C^*(E) \rightarrow \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ is the map defined in Theorem 5.13. More generally, since $\pi_{ap}(s_\alpha s_\alpha^*) = S_\alpha S_\alpha^* = R_\alpha$ then for every $X \in \Delta(E) = \overline{\text{span}}\{s_\mu s_\mu^* \mid \mu \in E^*\}$ and for all essentially aperiodic trails $\tau \in \mathcal{T}$, there exists a unique $c_\tau(X) \in \mathbb{C}$ such that

$$Q_\tau \pi_{ap}(X) = \pi_{ap}(X) Q_\tau = c_\tau(X) Q_\tau.$$

Furthermore, the projections $(Q_\tau)_{\tau \in \mathcal{T}}$ are mutually orthogonal and $\text{SOT} - \sum_{\tau \in \mathcal{T}} Q_\tau = I$.

Thus, for $X \in \Delta(E)$,

$$\pi_{ap}(X) = \pi_{ap}(X) \left(\text{SOT} - \sum_{\tau \in \mathcal{T}} Q_\tau \right) = \text{SOT} - \sum_{\tau \in \mathcal{T}} c_\tau(X) Q_\tau.$$

Lemma 5.17. *Suppose \mathcal{H} is an infinite, separable Hilbert space and consider a collection $\{P_n\}$ of mutually orthogonal projections such that $\text{SOT} - \sum_n P_n = I$. Then the map*

$$\begin{aligned} E : \mathcal{B}(\mathcal{H}) &\rightarrow \mathcal{B}(\mathcal{H}) \\ T &\mapsto \text{SOT} - \sum_n P_n T P_n \end{aligned}$$

is a conditional expectation of $\mathcal{B}(\mathcal{H})$ onto \mathcal{P}' where $\mathcal{P}' = \{T \in \mathcal{B}(\mathcal{H}) \mid TP_n = P_n T \ \forall n\}$.

Proof. It is immediately clear that E is positive, idempotent and satisfies $\text{ran } E \subseteq \mathcal{P}'$. If $T \in \mathcal{P}'$ then $E(T) = \text{SOT} - \sum_n P_n T P_n = \text{SOT} - \sum_n P_n T = T$, so $\text{ran } E = \mathcal{P}'$. Finally, for $T \in \mathcal{P}'$ and $S \in \mathcal{B}(\mathcal{H})$ we have

$$E(ST) = \text{SOT} - \sum_n P_n S T P_n = \text{SOT} - \sum_n P_n S P_n T = E(S)T,$$

so E is indeed a conditional expectation. □

Definition 5.18. Define the map $E_{ap} : \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z})) \rightarrow \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ by $E_{ap}(T) = \text{SOT} - \sum_{\tau \in \mathcal{T}} Q_\tau T Q_\tau$. The lemma above tells us that E_{ap} is a conditional expectation of $\mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$ onto $\mathcal{Q}' = \{T \in \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z})) \mid T Q_\tau = Q_\tau T \ \forall \tau \in \mathcal{T}\}$.

Now let $\mathcal{R} = \{R_\alpha = \pi_{ap}(s_\alpha s_\alpha^*)\}_{\alpha \in E^*}$. It's clear that $\mathcal{R} \subseteq \mathcal{Q}'$. Moreover, for some $T \in \mathcal{R}'$ and a fixed essentially aperiodic trail τ then $TR_{\tau(n)} = R_{\tau(n)}T$ so that

$$TQ_\tau = T \cdot \text{SOT} - \lim_n R_{\tau(n)} = \text{SOT} - \lim_n TR_{\tau(n)} = \text{SOT} - \lim_n R_{\tau(n)}T = Q_\tau T,$$

which implies $T \in \mathcal{Q}'$. Thus, for $T \in \mathcal{R}' \subseteq \mathcal{Q}'$ we have $E_{ap}(T) = T$.

5.2. The abelian core. Before we introduce the definition of the abelian core, we investigate the normal elements from our standard generating set $\{s_\mu s_\nu^* \mid \mu, \nu \in E^*\}$ of $C^*(E)$.

Proposition 5.19. *Suppose E is a row-finite graph with universal CK E -family $\{s, p\}$ which generates $C^*(E)$. Then a non-zero element of the form $X = s_\mu s_\nu^*$ with $\mu, \nu \in E^*$ is normal if and only if one of the following holds:*

- (1) $\mu = \nu$
- (2) $\mu = \nu \lambda^k$ for $k \geq 1$ where λ is an entryless cycle
- (3) $\nu = \mu \lambda^k$ for $k \geq 1$ where λ is an entryless cycle.

Moreover, such normal elements commute.

Proof. Take $X = s_\mu s_\nu^*$ with $\mu, \nu \in E^*$ to be an element of $C^*(E)$. If $\mu = \nu$ then clearly $X = X^*$, so X is also normal. Now suppose $\mu = \nu \lambda^k$ where λ is an entryless cycle. By Corollary 3.24, we know that $s_\lambda s_\lambda^* = p_{r(\lambda)} = p_{s(\lambda)}$ and it follows that $s_\lambda^k s_\lambda^{*k} = p_{r(\lambda)}$. In this case,

$$\begin{aligned} XX^* &= s_\mu s_\nu^* s_\nu s_\mu^* = s_\mu s_\mu^* = (s_\nu s_\lambda^k)(s_\lambda^{*k} s_\nu^*) = s_\nu s_\nu^* \\ X^* X &= s_\nu s_\mu^* s_\mu s_\nu^* = s_\nu s_\nu^*. \end{aligned}$$

Thus, in case (2), the element X is normal. A nearly identical argument will be valid for case (3).

Now suppose $X = s_\mu s_\nu^* \neq 0$ is normal. We know that $s(\mu) = s(\nu)$ and

$$s_\mu s_\mu^* = s_\mu s_\nu^* s_\nu s_\mu^* = XX^* = X^* X = s_\nu s_\mu^* s_\mu s_\nu^* = s_\nu s_\nu^*.$$

Since $s_\mu s_\mu^* \leq p_{r(\mu)}$ and $s_\nu s_\nu^* \leq p_{r(\nu)}$, then $r(\mu) = r(\nu)$. Moreover, X^2 is non-zero and thus, $s_\mu s_\nu^* s_\mu s_\nu^*$ is non-zero. Either $\mu = \nu \gamma$ or $\nu = \mu \gamma$ for some closed path γ . If $\mu = \nu$ then we're done. If not, suppose $\mu = \nu \gamma$ and that the closed path γ has an entry, so $s_\gamma s_\gamma^* < p_{r(\gamma)} = p_{s(\nu)}$. Then $XX^* = s_\nu s_\nu^* = s_\beta s_\gamma s_\gamma^* s_\nu^* < s_\nu s_\nu^* = X^* X$, a contradiction to the assumption that X is normal. Thus, γ cannot have an entry, and so it may be written uniquely as $\gamma = \lambda^k$ for some entryless cycle λ . The case where $\nu = \mu \gamma$ is similar.

We will now see that such elements commute. Suppose we are considering two elements of form (2) and (3), call them $s_\mu s_\nu^*$ and $s_\alpha s_\beta^*$ where $\mu = \nu \gamma^\ell$ and $\beta = \alpha \lambda^k$ and both λ and γ are entryless cycles. If we assume $s_\mu s_\nu^* s_\alpha s_\beta^*$ is non-zero, then either α extends ν or ν extends α . In either case, because both λ and γ do not have any entries, they must be equal and the difference between α and ν must be some multiple of $\lambda = \gamma$. Suppose $\alpha = \nu \gamma^m$, forcing $\beta = \nu \gamma^{k+m}$ and $\mu = \nu \gamma^{m+\ell}$. Then

$$\begin{aligned} s_\mu s_\nu^* s_\alpha s_\beta^* &= s_\nu s_\gamma^{m+\ell} s_\nu^* s_\nu s_\gamma^m s_\gamma^{*k+m} s_\nu^* = s_\nu s_\gamma^{2m+\ell} s_\gamma^{*k+m} s_\nu^* = s_\nu s_\gamma^{m+\ell-k} s_\nu^* \\ s_\alpha s_\beta^* s_\mu s_\nu^* &= s_\nu s_\gamma^m s_\gamma^{*k+m} s_\nu^* s_\nu s_\gamma^{m+\ell} s_\nu^* = s_\nu s_\gamma^{*m+\ell-k} s_\nu^*. \end{aligned}$$

Similarly, in the case where the two elements $s_\mu s_\nu^*$ and $s_\alpha s_\beta^*$ are of form either (2) or (3), we find that if $s_\mu s_\nu^* s_\alpha s_\beta^*$ is nonzero, then the two entryless cycles must be equal which forces the two elements to commute.

We are now left with the case where one element is of form (1) and the other is of form (2) (this will simultaneously give us the case where the two elements are of form (1) and (3) by taking adjoints). Suppose we have elements $s_\alpha s_\alpha^*$ and $s_\mu s_\nu^*$ where $\mu = \nu \lambda^k$ for an entryless cycle λ . Assume that $s_\alpha s_\alpha^* s_\mu s_\nu^*$ is nonzero, so either α extends μ or μ extends α .

If α extends μ , then there exists some ℓ such that $\alpha = \mu \lambda^\ell = \nu \lambda^{k+\ell}$ so that

$$s_\alpha s_\alpha^* s_\mu s_\nu^* = s_\nu (s_\lambda^{k+\ell} s_\lambda^{*k+\ell}) (s_\nu^* s_\nu) s_\lambda^k s_\nu^* = s_\nu s_\lambda^k s_\nu^* = s_\nu s_\lambda^k (s_\nu^* s_\nu) (s_\lambda^{k+\ell} s_\lambda^{*k+\ell}) s_\nu^* = s_\mu s_\nu^* s_\alpha s_\alpha^*.$$

Now, if μ extends α we have two subcases: either $\mu = \nu_1 \nu_2 \lambda^k$ where $\alpha = \nu_1$ and $\nu = \nu_1 \nu_2$ or $\mu = \nu \lambda^\ell \lambda_1 \lambda_2 \lambda^{k-\ell-1}$ where $\alpha = \nu \lambda^\ell \lambda_1$ and $\lambda = \lambda_1 \lambda_2$.

If the first is true, then

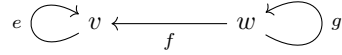
$$s_\alpha s_\alpha^* s_\mu s_\nu^* = s_{\nu_1} (s_{\nu_1}^* s_{\nu_1}) s_{\nu_2} s_\lambda^k s_{\nu_2}^* s_{\nu_1}^* = s_\nu s_\lambda^k s_\nu^* = s_{\nu_1} s_{\nu_2} s_\lambda^k s_{\nu_2}^* (s_{\nu_1}^* s_{\nu_1}) s_{\nu_1}^* = s_\mu s_\nu^* s_\alpha s_\alpha^*.$$

However, suppose now that the second subcase holds and $\mu = \nu \lambda^\ell \lambda_1 \lambda_2 \lambda^{k-\ell-1}$ where $\lambda = \lambda_1 \lambda_2$ and $\alpha = \nu \lambda^\ell \lambda_1$. Because λ is entryless, so is the path λ_1 , in that for every $v \in [\lambda_1]$ there is only one edge e with $r(e) = v$. Thus, $s_{\lambda_1} s_{\lambda_1}^* = p_{r(\lambda_1)}$, and so

$$s_\alpha s_\alpha^* s_\mu s_\nu^* = s_\nu s_\lambda^\ell s_{\lambda_1} s_{\lambda_1}^* s_\lambda^{*\ell} (s_\nu^* s_\nu) s_\lambda^k s_\nu^* = s_\nu s_\lambda^{\ell+k} s_\lambda^{*\ell} s_\nu^* = s_\nu s_\lambda^k (s_\nu^* s_\nu) s_\lambda^\ell (s_{\lambda_1} s_{\lambda_1}^*) s_\lambda^{*\ell} s_\nu^* = s_\mu s_\nu^* s_\alpha s_\alpha^*.$$

And so, despite requiring a number of subcases, we've found that the normal elements of the form $X = s_\mu s_\nu^*$ do indeed commute. \square

Example 5.20. We return to Graph 12 investigated previously in Section 3.2, shown again below.



All paths in the graph E have one of three forms, we can write

$$E^* = \{e^k f g^\ell \mid 0 \leq k, \ell\} \cup \{e^k \mid 0 \leq k\} \cup \{g^\ell \mid 0 \leq \ell\}.$$

Theorem 5.19, along with the fact that g is the only entryless cycle tells us that the normal elements of $C^*(E)$ will fall into one of the following sets:

$$\begin{aligned} & \{s_{e^k} s_{e^k}^* \mid 0 \leq k\} \\ & \{s_{e^k} s_f s_{g^\ell} s_{g^m}^* s_{g^\ell}^* s_f^* s_{e^k}^* \mid 0 \leq k, \ell, m\} \\ & \{s_{e^k} s_f s_{g^\ell} s_{g^m}^* s_{g^\ell}^* s_f^* s_{e^k}^* \mid 0 \leq k, \ell, m\} \\ & \{s_{g^k} s_{g^m}^* \mid 0 \leq k, m\} = \{s_{g^k} \mid 0 \leq k\} \cup \{s_{g^k}^* \mid 0 \leq k\}. \end{aligned}$$

Thus, we find that the only elements in the generating set $\{s_\mu s_\nu^* \mid \mu, \nu \in E^*\}$ which are not normal are those of the form

$$\begin{aligned} & \{s_{e^k} s_{e^\ell}^* \mid k \neq \ell\} \\ & \{s_{e^k} f g^\ell s_{g^m}^* = s_{e^k} f g^{\ell-m} \mid 0 \leq k, \ell, m\} \\ & \{s_{g^m} s_{e^k}^* f g^\ell = s_{e^k} f g^{\ell-m} \mid 0 \leq k, \ell, m\}. \end{aligned}$$

It is now easy to determine whether an element of the form $X = s_\mu s_\nu^*$ in $C^*(E)$ is normal in this example.

Definition 5.21. Let $\mathcal{M}(E)$ be the C^* -algebra generated by all such normal elements of the form $s_\mu s_\nu^*$. Proposition 5.19 implies that $\mathcal{M}(E)$ is abelian. Thus, we appropriately call $\mathcal{M}(E)$ the **abelian core** of $C^*(E)$.

Because $\mathcal{M}(E)$ consists of all elements $s_\alpha s_\alpha^*$ and $\mathcal{M}(E)$ is abelian, $\pi_{ap}(\mathcal{M}(E)) \subseteq \mathcal{R}' \subseteq \mathcal{Q}'$. Thus, for every element $X \in \mathcal{M}(E)$, we have $E_{ap}(\pi_{ap}(X)) = \pi_{ap}(X)$.

The following result will define a new conditional expectation which will allow us to relate the diagonal of E with the abelian core.

Theorem 5.22. *For a row-finite graph E , there exists a unique conditional expectation E_M of $C^*(E)$ onto $\mathcal{M}(E)$ such that*

$$E_M(X) = \begin{cases} X & \text{if } X = s_\alpha s_\beta^* \text{ is normal} \\ 0 & \text{if } X = s_\alpha s_\beta^* \text{ is not normal.} \end{cases}$$

Furthermore, $\pi_{ap} \circ E_M = E_{ap} \circ \pi_{ap}$ and in particular, E_M is faithful.

Proof. Our goal will be to verify that those $X = s_\alpha s_\beta^* \in C^*(E)$ which are not normal satisfy

$$E_{ap}(\pi_{ap}(X)) = E_{ap}(S_\alpha S_\beta^*) = \text{SOT} - \lim_{\tau \in \mathcal{T}} Q_\tau S_\alpha S_\beta^* Q_\tau = 0.$$

It is sufficient to show that for any essentially aperiodic trail τ , if $Q_\tau S_\alpha S_\beta^* Q_\tau$ is non-zero then $X = s_\alpha s_\beta^*$ is normal.

Note that

$$\begin{aligned} Q_\tau S_\alpha S_\beta^* Q_\tau \xi_\gamma^n &= Q_\tau S_\alpha S_\beta^* \begin{cases} \xi_\tau^n & \text{if } \gamma = \tau \\ 0 & \text{otherwise} \end{cases} \\ &= Q_\tau \begin{cases} \xi_{\alpha\tau'}^{n-|\beta|+|\alpha|} & \text{if } \gamma = \tau = \beta\tau' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \xi_\tau^{n-|\beta|+|\alpha|} & \text{if } \gamma = \tau = \beta\tau' = \alpha\tau' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus, if $Q_\tau S_\alpha S_\beta^* Q_\tau$ is non-zero, then $\tau = \beta\tau' = \alpha\tau'$, implying that either the two paths are equal, α extends β or β extends α . In the case where the two paths are equal, we immediately get that $X = s_\alpha s_\beta^*$ is normal. Let's suppose without loss of generality that α extends β , so $\alpha = \beta\lambda$, where $s(\alpha) = s(\beta)$ implies that λ is a closed path. Thus, $\alpha\lambda\tau' = \beta\tau' = \tau = \alpha\tau'$ and so since α is a finite path, we get that $\lambda\tau' = \tau'$. Thus, $\tau' = \lambda\lambda\tau' \dots$. We then have that τ is periodic with period λ and so, because τ is an essentially aperiodic trail, λ must be some multiple of an entryless cycle, call it ν . Thus, $\beta = \alpha\nu^k$ for the entryless cycle ν , making $X = s_\alpha s_\beta^*$ normal. A similar argument holds if α extends β .

Since $E_{ap}(\pi_{ap}(X)) = \pi_{ap}(X)$ for all normal $X = s_\alpha s_\beta^*$, then $E_{ap}(\pi_{ap}(C^*(E))) \subseteq \pi_{ap}(C^*(E))$. The injectivity of π_{ap} implies that we may set E_M to be $\pi_{ap}^{-1} \circ E_{ap} \circ \pi_{ap}$. This is a linear continuous map which satisfies $\pi_{ap} \circ E_M = E_{ap} \circ \pi_{ap}$. If \tilde{E}_M also satisfies this equation then $\pi_{ap} \circ \tilde{E}_M = E_{ap} \circ \pi_{ap} = \pi_{ap} \circ E_M$ and by applying π_{ap}^{-1} to both sides, we get $E_M = \tilde{E}_M$, giving us uniqueness of the operator E_M . Moreover, the injectivity of π_{ap} also tells us that

$$E_M(X) = \begin{cases} X & \text{if } X = s_\alpha s_\beta^* \text{ is normal} \\ 0 & \text{if } X = s_\alpha s_\beta^* \text{ is not normal.} \end{cases}$$

Finally, suppose we have some $X \in C^*(E)$ with $X \geq 0$ and $0 = E_M(X)$. Then $0 = \pi_{ap}(E_M(X)) = E_{ap}(\pi_{ap}(X)) = \pi_{ap}(X)$ and because π_{ap} is injective, then $0 = X$. Thus, by continuity and linearity, E_M is a faithful conditional expectation from $C^*(E)$ onto $\mathcal{M}(E)$. \square

We will now see the relationship between the abelian core and the diagonal.

Corollary 5.23. *Suppose E is a row-finite graph. Then*

$$\mathcal{M}(E) = \Delta(E)' = \{X \in C^*(E) \mid XD = DX \quad \forall D \in \Delta(E)\}.$$

Proof. We have seen previously that $\Delta(E) \subseteq \mathcal{M}(E)$ and that $\mathcal{M}(E)$ is abelian, so then $\mathcal{M}(E) \subseteq \mathcal{M}(E)' \subseteq \Delta(E)'$. We are only left to prove $\Delta(E)' \subseteq \mathcal{M}(E)$. Fix some $X \in \Delta(E)'$ so that for every path $\alpha \in E^*$,

$$\pi_{ap}(X)R_\alpha = \pi_{ap}(X s_\alpha s_\alpha^*) = \pi_{ap}(s_\alpha s_\alpha^* X) = R_\alpha \pi_{ap}(X).$$

Thus, $\pi_{ap}(X) \in \mathcal{R}'$, so $E_{ap}(\pi_{ap}(X)) = \pi_{ap}(X)$. By Theorem 5.22 we have $\pi_{ap}(X) = E_{ap}(\pi_{ap}(X)) = \pi_{ap}(E_M(X))$ and as π_{ap} is injective, $X = E_M(X) \in \mathcal{M}(E)$. \square

Definition 5.24. We define the map E_τ to be

$$\begin{aligned} E_\tau : C^*(E) &\rightarrow Q_\tau \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z})) Q_\tau \\ X &\mapsto Q_\tau \pi_{ap}(X) Q_\tau. \end{aligned}$$

For $X \in \mathcal{M}(E)$, the fact that $E_M(X) = X$ implies that

$$\pi_{ap}(X) = E_{ap}(\pi_{ap}(X)) = \text{SOT} - \sum_{\tau \in \mathcal{T}} Q_\tau \pi_{ap}(X) Q_\tau = \text{SOT} - \sum_{\tau \in \mathcal{T}} E_\tau(X) = \text{SOT} - \bigoplus_{\tau \in \mathcal{T}} E_\tau(X).$$

Note that the above equality combined with the fact that π_{ap} is a *-homomorphism tells us that restricting E_τ to $\mathcal{M}(E)$ gives a *-homomorphism. Let

$$\pi_\tau^\mathcal{M} := E_\tau|_{\mathcal{M}(E)} : \mathcal{M}(E) \rightarrow Q_\tau \mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z})) Q_\tau.$$

Additionally, the system $(E_\tau)_{\tau \in \mathcal{T}}$ is **jointly faithful** in that for $X \in C^*(E)$ with $X \geq 0$ and $E_\tau(X) = 0$ for all $\tau \in \mathcal{T}$ then $X = 0$. Indeed, for a non-zero $X \in C^*(E)$, injectivity of π_{ap} tells us that $\pi_{ap}(X^*X)$ is a non-zero element of $\mathcal{B}(\ell^2(\mathcal{T} \times \mathbb{Z}))$. Suppose we write $\pi_{ap}(X^*X)$ as $\sum c_{\gamma,n} \xi_\tau^n$ where there exists at least one $(\gamma, n) \in \mathcal{T} \times \mathbb{Z}$ where the corresponding constant $c_{\gamma,n} \neq 0$. Then $E_\tau(\pi_{ap}(X^*X)) = Q_\gamma \pi_{ap}(X^*X) Q_\gamma = c_{\gamma,n} \xi_\gamma^n \neq 0$. Thus, the collection $(E_\tau)_{\tau \in \mathcal{T}}$ is indeed jointly faithful.

Proposition 5.25. *Suppose E is a row-finite graph. If $\tau \in \mathcal{T}$ is either finite or continuous, then there exists a unique state ω_τ on $C^*(E)$ such that $E_\tau(X) = \omega_\tau(X) Q_\tau$ for all $X \in C^*(E)$.*

Proof. We first note that

$$E_{ap}(E_\tau(X)) = \text{SOT} - \sum_{\gamma \in \mathcal{T}} Q_\gamma E_\tau(X) Q_\gamma = \text{SOT} - \sum_{\gamma \in \mathcal{T}} Q_\gamma Q_\tau \pi_{ap}(X) Q_\tau Q_\gamma = Q_\tau \pi_{ap}(X) Q_\tau = E_\tau(X),$$

and

$$E_{ap}(E_\tau(X)) = E_{ap}(Q_\tau \pi_{ap}(X) Q_\tau) = Q_\tau E_{ap}(\pi_{ap}(X)) Q_\tau = Q_\tau \pi_{ap}(E_M(X)) Q_\tau.$$

Together, these imply that $E_\tau(X) = Q_\tau \pi_{ap}(E_M(X)) Q_\tau$. Because $E_M(X) = 0$ for $X = s_\alpha s_\beta^*$ which are not normal, we only need to confirm the equation for normal generators $X = s_\alpha s_\beta^*$.

Consider the normal element $X = s_\alpha s_\beta^*$ with the additional assumption that $\alpha \neq \beta$. Proposition 5.19 implies that either α or β must begin with an entryless cycle. Suppose α is the path which begins with an entryless cycle λ . The assumption that τ is not periodic tells us that if τ extended α , τ would be forced to be periodic. If we assumed α extended τ , then this would force τ to be finite and $s(\tau)$ would not be a source in this case. In either of these assumptions, we reach a contradiction. Thus, $s_\alpha^* s_{\tau(n)} s_{\tau(n)}^* = 0$ for $n \geq |\alpha|$.

Similarly, if we assume β begins with an entryless cycle, $s_\beta^* s_{\tau(n)} s_{\tau(n)}^* = 0$ for $n \geq |\beta|$. In either case,

$$s_{\tau(n)} s_{\tau(n)}^* s_\alpha s_\beta^* s_{\tau(n)} s_{\tau(n)}^* = 0 \quad \forall n \geq \max\{|\alpha|, |\beta|\}.$$

Thus, $R_{\tau(n)} S_\alpha S_\beta^* R_{\tau(n)} = 0$ and so

$$\begin{aligned} E_\tau(s_\alpha s_\beta^*) &= Q_\tau S_\alpha S_\beta^* Q_\tau \\ &= \left(\text{SOT} - \lim_n R_{\tau(n)} \right) S_\alpha S_\beta^* \left(\text{SOT} - \lim_m R_{\tau(m)} \right) \\ &= 0. \end{aligned}$$

We are now left to check the normal elements $X = s_\alpha s_\alpha^*$. In this case, we find that $E_\tau(X)$ is equal to Q_τ if $\tau = \alpha\tau'$ and 0 otherwise.

□

Note that it immediately follows from the proof that $\omega_\tau = \omega_\tau \circ E_M$.

Lemma 5.26. *Suppose E is a row-finite graph. For every element $X = s_\alpha s_\beta^* \in C^*(E)$ and every continuous $\tau \in \mathcal{T}$, there exists some $N_\tau \geq 0$ such that for all $n \geq N_\tau$ we have*

$$s_{\tau(n)} s_{\tau(n)}^* X s_{\tau(n)} s_{\tau(n)}^* = \omega_\tau(X) s_{\tau(n)} s_{\tau(n)}^*.$$

Proof. By taking adjoints if necessary, we may assume without loss of generality that $|\alpha| \geq |\beta|$. Suppose we choose $n > |\alpha|$ and assume $s_{\tau(n)} s_{\tau(n)}^* X s_{\tau(n)} s_{\tau(n)}^* \neq 0$. This immediately tells us that $s_\alpha s_\beta^*$, $s_{\tau(n)}^* s_\alpha$, and $s_\beta^* s_{\tau(n)}$ are non-zero. Our assumptions on length imply that $\tau(n)$ must extend both α and β . Thus, $\beta \preceq \alpha \prec \tau(n)$. Let's write $\alpha = \beta\lambda$ where the assumption that $s(\alpha) = s(\beta)$ tells us that λ is some closed path.

If $\alpha \neq \beta$ then $|\lambda| \geq 1$ and because τ is not periodic, there exists a maximal k such that $\beta\lambda^k \prec \tau$. Let $N_\tau = |\beta| + (k+1)|\lambda| > |\alpha|$ and choose any $n \geq N_\tau$. Then $\tau(n) = \beta\lambda^k\gamma$ for some γ , and so

$$\begin{aligned} s_{\tau(n)}^* s_\alpha s_\beta^* s_{\tau(n)} &= s_\gamma^* s_{\lambda^k}^* s_\beta^* s_\beta s_\lambda s_\beta^* s_\beta s_{\lambda^k} s_\gamma \\ &= s_\gamma^* s_{\lambda^k}^* s_{\lambda^{k+1}} s_\gamma && (\text{since } s_\beta^* s_\beta = p_{s(\beta)}) \\ &= s_\gamma^* s_\lambda s_\gamma. \end{aligned}$$

If $s_\gamma^* s_\lambda \neq 0$ then since $|\gamma| = |\tau(n)| - |\beta\lambda^k| \geq (|\beta| + (k+1)|\lambda|) - (|\beta| + k|\lambda|) = |\lambda|$, we have that $\lambda \prec \gamma$. But this contradicts the maximal choice of k .

Thus, if $s_{\tau(n)} s_{\tau(n)}^* s_\alpha s_\beta^* s_{\tau(n)} s_{\tau(n)}^* \neq 0$ for $n > |\alpha|$ then $\alpha = \beta$. We may now take $N_\tau = |\alpha|$. For $n \geq N_\tau$, if we write $\tau(n) = \alpha\gamma$ then $s_{\tau(n)} s_{\tau(n)}^* s_\alpha s_\beta^* s_{\tau(n)} s_{\tau(n)}^* = s_{\tau(n)} s_{\tau(n)}^*$. \square

Proposition 5.27. *Suppose E is a row-finite graph. If $\tau \in \mathcal{T}$ is continuous, then for $X \in C^*(E)$ and $\omega \in \mathbb{C}$, the following are equivalent:*

- (1) $\lim_n \|s_{\tau(n)} s_{\tau(n)}^* X s_{\tau(n)} s_{\tau(n)}^* - \omega s_{\tau(n)} s_{\tau(n)}^*\| = 0$
- (2) $\lim_n \|E_M(X) s_{\tau(n)} s_{\tau(n)}^* - \omega s_{\tau(n)} s_{\tau(n)}^*\| = 0$
- (3) $\omega = \omega_\tau(X)$.

Proof. (3) \Rightarrow (1) It suffices to prove this for elements of the form $X = s_\alpha s_\beta^*$. This follows directly from Lemma 5.26.

(1) \Rightarrow (2) Assume (1) holds. By applying E_M to (1), we get that

$$\lim_n \|E_M(s_{\tau(n)} s_{\tau(n)}^* X s_{\tau(n)} s_{\tau(n)}^*) - \omega E_M(s_{\tau(n)} s_{\tau(n)}^*)\| = 0.$$

Thus implying $\lim_n \|s_{\tau(n)} s_{\tau(n)}^* E_M(X) s_{\tau(n)} s_{\tau(n)}^* - \omega s_{\tau(n)} s_{\tau(n)}^*\| = 0$. Since we know $E_M(X)$ commutes with $s_{\tau(n)} s_{\tau(n)}^*$, then

$$\lim_n \|E_M(X) s_{\tau(n)} s_{\tau(n)}^* - \omega s_{\tau(n)} s_{\tau(n)}^*\| = 0.$$

(2) \Rightarrow (3) Assuming (2) and applying π_{ap} , we find that

$$\lim_n \|\pi_{ap}(E_M(X)s_{\tau(n)}s_{\tau(n)}^*) - \omega\pi_{ap}(s_{\tau(n)}s_{\tau(n)}^*)\| = 0.$$

We may rewrite this as $\lim_n \|E_{ap}(\pi_{ap}(X))R_{\tau(n)} - \omega R_{\tau(n)}\| = 0$. This implies that $\text{SOT-}\lim_n E_{ap}(\pi_{ap}(X))R_{\tau(n)} - \omega R_{\tau(n)} = 0$. Rearranging, we find $E_{ap}(\pi_{ap}(X))Q_\tau = \omega Q_\tau$.

We also know that $E_{ap}(\pi_{ap}(X))Q_\tau = Q_\tau E_{ap}(\pi_{ap}(X))Q_\tau = E_\tau(X) = \omega_\tau(X)Q_\tau$ where the final equality holds from Proposition 5.25. Thus, $\omega_\tau(X) = \omega$. \square

Definition 5.28. For a trail τ which is discrete, we define the **essential path** of τ to be

$$\tau_{ess} = \begin{cases} \tau & \text{if } \tau \text{ is finite} \\ \alpha & \text{if } \tau \text{ is periodic with } \tau = \alpha\lambda\lambda\lambda\ldots \text{ and } |\alpha| \text{ minimal.} \end{cases}$$

Note that we may recover the discrete trail τ from τ_{ess} . Indeed, if $s(\tau_{ess})$ is a source, then $\tau = \tau_{ess}$ and if it is not a source, then there will exist a unique entryless cycle λ with $r(\lambda) = s(\lambda) = s(\tau_{ess})$ and so we recover τ as $\tau_{ess}\lambda\lambda\ldots$.

If τ is discrete, the sequence $(R_{\tau(n)})$ eventually becomes constant and equal to $R_{\tau_{ess}}$ and so because $\text{SOT-}\lim_n R_{\tau(n)} = Q_\tau$, then $R_{\tau_{ess}} = Q_\tau$. Now note that we have the following equalities:

$$\begin{aligned} \pi_{ap}(E_M(X)s_{\tau_{ess}}s_{\tau_{ess}}^*) &= \pi_{ap}(E_M(X))Q_\tau \\ &= E_{ap}(\pi_{ap}(X))Q_\tau \\ &= \text{SOT-}\sum_{\gamma \in \mathcal{T}} Q_\gamma \pi_{ap}(X) Q_\gamma Q_\tau \\ &= Q_\tau \pi_{ap}(X) Q_\tau \\ &= E_\tau(X) \\ &= R_{\tau_{ess}} \pi_{ap}(X) R_{\tau_{ess}} \\ &= \pi_{ap}(s_{\tau_{ess}}s_{\tau_{ess}}^* X s_{\tau_{ess}}s_{\tau_{ess}}^*). \end{aligned}$$

Since π_{ap} is injective, then

$$E_M(X)s_{\tau_{ess}}s_{\tau_{ess}}^* = s_{\tau_{ess}}s_{\tau_{ess}}^* X s_{\tau_{ess}}s_{\tau_{ess}}^*.$$

Definition 5.29. For a discrete trail $\tau \in \mathcal{T}$, we define $\mathcal{M}_\tau(E) = s_{\tau_{ess}}s_{\tau_{ess}}^* C^*(E)s_{\tau_{ess}}s_{\tau_{ess}}^*$. If τ is finite, then we get that $\mathcal{M}_\tau(E) = \mathbb{C}s_\tau s_\tau^* = \mathbb{C}s_{\tau_{ess}}s_{\tau_{ess}}^*$. On the other hand, if τ is infinite and $\tau = \alpha\lambda\lambda\lambda\ldots$ where λ is an entryless cycle, then $\tau_{ess} = \alpha$ and $\mathcal{M}_\tau(E) = C^*(s_\alpha s_\lambda s_\alpha^*)$.

Let the map F_τ be given by

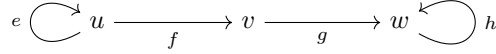
$$\begin{aligned} F_\tau : C^*(E) &\rightarrow \mathcal{M}_\tau(E) \\ X &\mapsto s_{\tau_{ess}}s_{\tau_{ess}}^* X s_{\tau_{ess}}s_{\tau_{ess}}^*. \end{aligned}$$

Thus, $F_\tau(X) = s_{\tau_{ess}} s_{\tau_{ess}}^* X s_{\tau_{ess}} s_{\tau_{ess}}^* = E_M(X) s_{\tau_{ess}} s_{\tau_{ess}}^*$ which tells us that $\pi_{ap}(F_\tau(X)) = E_\tau(X)$.

Definition 5.30. We call a path α **distinguished** if there exists a unique entryless cycle λ_α with $r(\lambda_\alpha) = s(\lambda_\alpha) = s(\alpha)$. For a distinguished path α and the unique corresponding cycle λ_α , let $w_\alpha = s_\alpha s_{\lambda_\alpha} s_\alpha^*$. Note that w_α is a normal partial isometry.

Moreover, if τ is an infinite, discrete essentially aperiodic trail, then τ_{ess} is distinguished.

Example 5.31. Consider Graph 28.



GRAPH 28

The paths f , gf , and $h^k gf$ for $k \in \mathbb{N}$ are all distinguished with $\lambda_\alpha = e$. Note that the paths e^ℓ , $f e^\ell$, $g f e^\ell$, and $h^k g f e^\ell$ for $k, \ell \in \mathbb{N}$ are also distinguished paths with $\lambda_\alpha = e$. No other paths in the graph are distinguished.

We will use the following result in our final theorem.

Proposition 5.32. Suppose $\mathfrak{A}, \mathfrak{B}$ are C^* -algebras with $\mathfrak{A} \subseteq \mathfrak{B}$. Then for any $a \in \mathfrak{A}$, we have $\text{spec}_{\mathfrak{B}}(a) \cup \{0\} = \text{spec}_{\mathfrak{A}}(a) \cup \{0\}$.

We are now ready to present the final theorem of this paper.

Theorem 5.33. Suppose E is a row-finite graph. For a $*$ -homomorphism $\Phi : C^*(E) \rightarrow \mathfrak{A}$, the following are equivalent:

- (1) Φ is injective
- (2) Φ restricted to $\mathcal{M}(E)$ is injective
- (3) both of the following conditions are satisfied:
 - (a) $\Phi(s_\alpha s_\alpha^*) \neq 0$ for all paths $\alpha \in E^*$
 - (b) for all distinguished paths $\alpha \in E^*$, $\text{spec}_{\mathfrak{A}}(\Phi(w_\alpha)) \supseteq \mathbb{T}$.

Proof. (1) \Rightarrow (2) This is clear.

(2) \Rightarrow (3) Because $s_\alpha s_\alpha^* \in \mathcal{M}(E)$ for all paths $\alpha \in E^*$, it's immediately clear that (a) holds. To see that (b) holds, fix some distinguished path $\alpha \in E^*$ and consider the normal partial isometry w_α . We have that $(w_\alpha w_\alpha^*) w_\alpha = w_\alpha$ and $(w_\alpha w_\alpha^*) w_\alpha^* = w_\alpha^* w_\alpha w_\alpha^* = w_\alpha^*$. Similarly, $w_\alpha^* (w_\alpha w_\alpha^*) = w_\alpha^*$ and $w_\alpha (w_\alpha w_\alpha^*) = w_\alpha$. Thus, $w_\alpha w_\alpha^*$ acts as the identity on w_α and w_α^* and so acts as the identity on all of $C^*(w_\alpha)$. This tells us precisely that w_α is unitary on $C^*(w_\alpha)$ and so $\text{spec}_{C^*(w_\alpha)}(w_\alpha) \subseteq \mathbb{T}$.

We know that there exists a gauge action on $C^*(E)$ which for any $z \in \mathbb{T}$, satisfies $\gamma_z(w_\alpha) = z^{|\lambda_\alpha|} w_\alpha$ and $\gamma_z(w_\alpha^*) = z^{-|\lambda_\alpha|} w_\alpha^*$. Thus, $\gamma_z(C^*(w_\alpha)) \subseteq C^*(w_\alpha)$ for all $z \in \mathbb{T}$ and so we have the action $\gamma : \mathbb{T} \rightarrow \text{Aut}(C^*(w_\alpha))$ which sends w_α to $z^{|\lambda_\alpha|} w_\alpha$. Thus, if $\text{spec}_{C^*(w_\alpha)}(w_\alpha) \subsetneq \mathbb{T}$, we may find some z with $|z| = 1$ and $z \notin \text{spec}_{C^*(w_\alpha)}(w_\alpha)$.

Then for any $x \in \mathbb{T}$, we have that

$$1 = \gamma_x(1) = \gamma_x((z1 - w_\alpha)^{-1} x^{|\lambda_\alpha|} \bar{x}^{|\lambda_\alpha|} (z1 - w_\alpha)) = \gamma_x((z1 - w_\alpha)^{-1} x^{|\lambda_\alpha|}) (\bar{x}^{|\lambda_\alpha|} z1 - w_\alpha).$$

This tells us that $\bar{x}^{|\lambda_\alpha|}z1 - w_\alpha$ is invertible and so $\bar{x}^{|\lambda_\alpha|}z \notin \text{spec}_{C^*(w_\alpha)}(w_\alpha)$. By choosing appropriate $x \in \mathbb{T}$, we can show that $\text{spec}_{C^*(w_\alpha)}(w_\alpha)$ does not contain any element from \mathbb{T} . However, the spectrum is non-empty and so our assumption that $\text{spec}_{C^*(w_\alpha)}(w_\alpha) \subsetneq \mathbb{T}$ must be incorrect. Thus, $\text{spec}_{C^*(w_\alpha)}(w_\alpha) = \mathbb{T}$.

Because $\mathcal{M}(E)$ is C^* -subalgebra of $C^*(E)$ with $w_\alpha \in \mathcal{M}(E)$, then $\text{spec}_{C^*(E)}(w_\alpha) \cup \{0\} = \text{spec}_{\mathcal{M}(E)}(w_\alpha) \cup \{0\}$ by Proposition 5.32. Except in the trivial case where w_α is an isometry, we have that $\text{spec}_{C^*(E)}(w_\alpha) = \text{spec}_{\mathcal{M}(E)}(w_\alpha) = \mathbb{T} \cup \{0\}$. Because $\Phi(\mathcal{M}(E))$ is isometrically isomorphic to $\mathcal{M}(E)$, we also get $\text{spec}_{\Phi(\mathcal{M}(E))}(\Phi(a)) = \text{spec}_{\mathcal{M}(E)}(a)$. Putting these pieces together, we find that $\mathbb{T} \subseteq \text{spec}_{\mathcal{M}(E)}(a) = \text{spec}_{\Phi(\mathcal{M}(E))}(\Phi(a)) = \text{spec}_{\mathfrak{A}}(\Phi(w_\alpha))$, where the last equality holds by Proposition 5.32 and the fact that $\Phi(\mathcal{M}(E))$ is a C^* -subalgebra of \mathfrak{A} .

(3) \Rightarrow (1) Fix some element $X \in \ker \Phi$ and suppose first that $\tau \in \mathcal{T}$ is discrete. Then it follows that $s_{\tau_{ess}} s_{\tau_{ess}}^* X^* X s_{\tau_{ess}} s_{\tau_{ess}}^* = F_\tau(X^* X) \in \ker \Phi \cap \mathcal{M}_\tau(E)$. Now, $\mathbb{T} \subseteq \text{spec}_{\mathfrak{A}}(\Phi(w_\alpha))$ implies that $\Phi(w_\alpha) \neq 0$. Moreover, (a) tells us that $\Phi(s_\alpha s_\alpha^*) \neq 0$ combined with the fact that given a discrete $\tau \in \mathcal{T}$, τ_{ess} is a distinguished path, then Φ is injective on the generators of $\mathcal{M}_\tau(E)$. Thus, in a proof similar to that shown in Proposition 5.12, Φ is also injective on all of $\mathcal{M}_\tau(E)$. Thus, $F_\tau(X^* X) = 0$. Now, we may utilize that $E_\tau(X) = \pi_{ap}(F_\tau(X)) = 0$.

In the case that $\tau \in \mathcal{T}$ is continuous, then by the result of Proposition 5.27, we have that

$$\lim_n \|s_{\tau(n)} s_{\tau(n)}^* X^* X s_{\tau(n)} s_{\tau(n)}^* - \omega_\tau(X^* X) s_{\tau(n)} s_{\tau(n)}^*\| = 0.$$

By applying Φ to this result and recalling that $\Phi(X^* X) = 0$, we get that

$$|\omega_\tau(X^* X)| \left(\lim_n \|\Phi(s_{\tau(n)} s_{\tau(n)}^*)\| \right) = \lim_n \|\omega_\tau(X^* X) \Phi(s_{\tau(n)} s_{\tau(n)}^*)\| = 0.$$

Now note that assumption (3) (a) combined with

$$\|\Phi(s_{\tau(n)} s_{\tau(n)}^*)\| = \|\Phi(s_{\tau(n)} s_{\tau(n)}^* s_{\tau(n)} s_{\tau(n)}^*)\| = \|\Phi(s_{\tau(n)} s_{\tau(n)}^*) \Phi(s_{\tau(n)} s_{\tau(n)}^*)^*\| = \|\Phi(s_{\tau(n)} s_{\tau(n)}^*)\|^2$$

tells us that $\|\Phi(s_{\tau(n)} s_{\tau(n)}^*)\| = 1$ for all $n \geq 1$. Thus, we must have $\omega_\tau(X^* X) = 0$ and so $E_\tau(X^* X) = \omega_\tau(X^* X) Q_\tau = 0$. We have now shown for that for any $\tau \in \mathcal{T}$, $E_\tau(X^* X) = 0$. Finally, because the collection $\{E_\tau\}_{\tau \in \mathcal{T}}$ is jointly continuous, we have that $X^* X = 0$. Equivalently, we have that $\ker \Phi = \{0\}$. \square

REFERENCES

- [1] J. Conway, A Course in Functional Analysis, Graduate Texts in Mathematics, **2**. (1990). Springer-Verlag, New York, NY.
- [2] J. Conway, A Course in Operator Theory, Graduate Texts in Mathematics, **21**. (2000). Springer-Verlag, New York, NY.
- [3] J. Cuntz, Simple C^* -Algebras Generated by isometries, Comm. Math. Phys. **57**. (1977). pp. 173-185.
- [4] J. Cuntz, W. Krieger, A class of C^* -algebras and topological Markov chains, Invent. Math. **56**. (1980). pp. 251-268.
- [5] K. Davidson, C^* -algebras by example, Fields Institute Monographs, **6**. (1996). American Mathematical Society, Providence, RI.
- [6] P. Fillmore, J. Mingo, Operator Algebras and Their Applications, Fields Institute Communications, **13**. (1994). American Mathematical Society, Providence, RI.
- [7] E. Hossain, The Algebraic Kirchberg-Phillips Conjecture for Leavitt Path Algebras. (2015). Available at: <https://uwspace.uwaterloo.ca/bitstream/handle/10012/9676/Hossain.Ehsaan.pdf?sequence=1>.
- [8] G. Murphy, C^* -Algebras and Operator Theory. (1990). Academic Press, Inc., Boston, MA.
- [9] G. Nagy, S. Reznikoff, Abelian core of graph algebras. **2**. (2012). London Mathematical Society. pp. 889-908.
- [10] I. Raeburn, Graph algebras, CBMS Regional Conference Series in Mathematics, **103**. (2005). American Mathematical Society, Providence, RI.
- [11] R. Sanders, Graph C^* -algebras. (2008). Available at: <http://www.waltervansuijlekom.nl/wp-content/uploads/2014/05/thesisRichardBSc.pdf>.
- [12] W. Szymański, General Cuntz-Krieger uniqueness theorem. **13**. (2002). Internat. J. Math.. pp. 549-555.
- [13] S.L. Woronowicz, Twisted $SU(2)$ Group. An Example of a Non-Commutative Differential Calculus. (1987). **23**. Publications of the Research Institute for Mathematical Sciences. pp. 117-181.